Critical thresholds in Euler-Poisson-alignment systems

Manas Bhatnagar
Iowa State University (USA)
(with Hailiang Liu and Changhui Tan)

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What's the question?

- Time regularity for convection-dominated problems,

$$\partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla_x)\mathbf{u} = \mathbf{F}, \quad x \in \mathbb{R}^n,$$

  to see whether they admit

– Global smooth solution

– Develops shocks/aggregates

– *Critical threshold* phenomena

global regularity $\sim$ a set of initial configurations
shocks/aggregation $\sim$ another set of initial configurations

E. Tadmor, S. Engelberg, H. Liu, C. Tan, Y. Lee T. Li ... are few of the many researchers interested, working and having significant contributions in this area

Motivation to CTP

Pressureless Euler equations with damping force, $\lambda > 0$,

$$
\begin{aligned}
&\rho_t + (\rho u)_x = 0, \quad (t, x) \in (0, \infty) \times \mathbb{R}, \\
&u_t + u u_x = -\lambda u,
\end{aligned}
$$

with $\rho(0, x) = \rho_0(x), u(0, x) = u_0(x)$.

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$$u_t + uu_x = -\lambda u,$$

with $\rho(0,x) = \rho_0(x), u(0,x) = u_0(x)$. Spatial derivative of damped Burgers':

$$(u_x)_t + u(u_x)_x = -u_x^2.$$

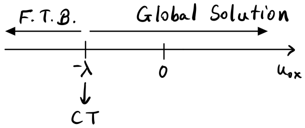
Using method of characteristics,

$$u_x(t, X(t;\alpha)) = \frac{\lambda}{e^{\lambda t}\left((\lambda/u_{0x}(\alpha)) + 1\right) - 1}, \quad \alpha \in \mathbb{R}.$$

Hence, if for some x^*, $u_{0x}(x^*) < -\lambda$, then

$$\lim_{t \to t^{*-}} u_x = -\infty, \quad t^* = \frac{1}{\lambda} \ln\left(\frac{u_{0x}(x^*)}{\lambda + u_{0x}(x^*)}\right).$$

u_x is bounded for all time if and only if $u_{0x}(x) \geq -\lambda$ for all $x \in \mathbb{R}$.



Criteria of breakdown: $u_x \to -\infty$ (shock formation) along with $\rho \to \infty$ (density concentration) at the same time.

EPA system

A class of 1D Euler-Poisson-alignment equations,

$$\rho_t + (\rho u)_x = 0, \quad x \in \mathbb{T} \quad \text{(Mass conservation)}$$
$$u_t + uu_x = -k\phi_x + \psi * (\rho u) - u\psi * \rho, \quad \text{(Momentum)}$$
$$-\phi_{xx} = \rho - c(x), \quad \text{(Poisson equation)}$$

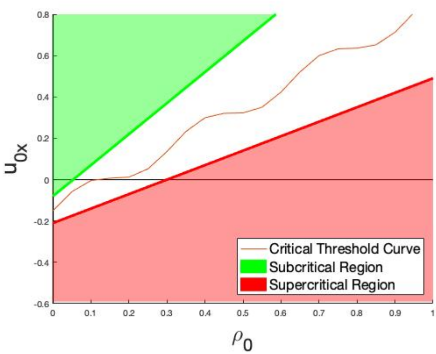
with initial conditions $(\rho_0(x) > 0, u_0(x)) \in H^s(\mathbb{R}) \times H^{s+1}(\mathbb{R})$, $s > 3/2$.

- ψ is the influence function modelling the mid-range alignment forces
- c is the background term which, in practicality, depends on the spatial variable but is most often assumed a constant. Need $\int_{\mathbb{T}} \rho_0 - c\, dx = 0$
- k signifies property of underlying forcing: repulsive ($k > 0$) or attractive ($k < 0$).
- Mix of Euler-alignment ($k = 0$) and Euler-Poisson system ($\psi \equiv 0$)

$c = c(x)$, $\psi \equiv 0$ and $k < 0$

$$\rho_t + (\rho u)_x = 0, \quad x \in \mathbb{T},$$
$$u_t + uu_x + \nu u = -k\phi_x,$$
$$-\phi_{xx} = \rho - c(x),$$

with $\rho_0 \geq 0, u_0 \in H^s(\mathbb{T}) \times H^{s+1}(\mathbb{T})$, $s > 3/2$.



Existing research

- $\psi \equiv 0$: Engelberg et al. (*Indiana Univ Math J*, 2001) (EP); MB, Liu (*M3AS*, 2020) (EP)
- $\psi \in L^\infty$: Tadmor and Tan (*PTRS-A*, 2014) (EA); Carrillo et al. (*M3AS*, 2016) (EPA)
- Strongly singular ($\psi \sim r^{-1-\epsilon}$): Do et al. (*ARMA*, 2017) (EA); Kiselev and Tan (*SIMA*, 2018) (EPA)
- $\psi \in L^1$ (weakly singular EA): Tan (*Nonlinearity*, 2020); MB, Liu (*AML*, 2022)
- $\psi \in L^\infty$ ($c = c(x), k < 0$): MB, Liu (*PhysicaD*, 2020) (EP/EPA)
- $\psi \in L^1, L^\infty$ ($k > 0$): MB, Liu, Tan (EPA)

Model

EPA system with $k > 0$:

$$\rho_t + (\rho u)_x = 0, \quad x \in \mathbb{T},$$
$$u_t + uu_x = -k\phi_x + \psi * (\rho u) - u\psi * \rho,$$
$$-\phi_{xx} = \rho - \bar{\rho}, \quad \bar{\rho} = \int_{\mathbb{T}} \rho_0(x)\, dx,$$

with initial data $\rho_0(x), u_0(x)$.

- $\psi \in L^\infty_+(\mathbb{T})$ and symmetric

Equivalent system (ρ, G):

$$\rho_t + (\rho u)_x = 0,$$
$$G_t + (Gu)_x = k(\rho - \bar{\rho}),$$
$$u_x = G - \psi * \rho,$$

with initial data $\rho_0(x), G_0(x) := u_{0x}(x) + \psi * \rho_0(x)$.

Theorem (B-Liu-Tan)

Let $k > 0$, $\psi \geq 0$ be bounded and $2\sqrt{k/\bar{\rho}} > \max \psi$. If

$$\max \psi - \min \psi < \frac{e^{\frac{\tan^{-1}\hat{z}}{\hat{z}}} \left(1 - e^{-\frac{\pi}{\tilde{z}} - \frac{\pi}{\hat{z}}}\right) \sqrt{k/\bar{\rho}}}{\left(1 + e^{-\frac{\pi}{\tilde{z}}}\right)},$$

then there exists a set Σ such that if

$$(G_0(x), \rho_0(x)) \in \Sigma, \quad \forall x \in \mathbb{T},$$

then there is global smooth solution. Here,

$$\tilde{z} := \sqrt{(4k/\bar{\rho} \min \psi^2) - 1} \qquad \hat{z} := \sqrt{(4k/\bar{\rho} \max \psi^2) - 1}.$$

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then there is global smooth solution. Here,

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Theorem (B-Liu-Tan)

Let $k > 0$, $\psi \geq 0$ be bounded and $2\sqrt{k/\bar{\rho}} \leq \min \psi$. There exists a set Σ_2 such that if

$$(G_0(x), \rho_0(x)) \in \Sigma_2, \quad \forall x \in \mathbb{T},$$

then there is global smooth solution.

Sketch of proof

- Construct a region using segments of trajectory curves from an auxiliary system.
- Show that the constructed region is invariant.

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- Show that the constructed region is invariant.

With $w := \frac{G}{\rho}$ and $s := \frac{1}{\rho}$:

$$w' = k - k\bar{\rho}s,$$
$$s' = w - (\psi * \rho)s.$$

Ensure $s(t) > 0$ for global solution. Alternatively, if $s(t_c) = 0$ then $\lim_{t \to t_c^-} \rho(t, x_c) = \infty$.

$\psi_{max}^2 < 4k/\bar{\rho}$

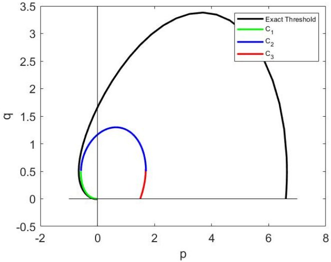


Figure: Invariant region.

$$\hat{p}' = k - k\bar{\rho}\hat{q}, \quad \hat{p}(0) = 0,$$
$$\hat{q}' = \hat{p} - \bar{\rho}\psi_{max} q, \quad \hat{q}(0) = 0.$$

Trace the backwards trajectory until $t_1 < 0$ such that $\hat{q}(t_1) = 1/\bar{\rho}$. Let $p_1 = \hat{p}(t_1) < 0$.

$\psi_{max}^2 < 4k/\bar{\rho}$

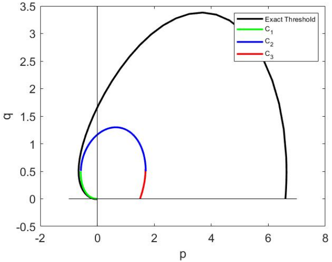


Figure: Invariant region.

$$\tilde{p}' = k - k\bar{\rho}\tilde{q}, \quad \hat{p}(0) = p_1,$$
$$\tilde{q}' = \tilde{p} - \bar{\rho}\psi_{min} q, \quad \hat{q}(0) = 1/\bar{\rho}.$$

Trace the backwards trajectory until $t_2 < 0$ such that $\tilde{q}(t_2) = 1/\bar{\rho}$ again. Let $p_2 = \tilde{p}(t_2) > 0$.

Invariance of the Region

$$w' = k - k\bar{\rho}s,$$
$$s' = w - (\psi * \rho)s.$$

$$\hat{p}' = k - k\bar{\rho}\hat{q},$$
$$\hat{q}' = \hat{p} - \beta_{max}q.$$

Proposition

$(w(0), s(0)) \in \bar{\Sigma} \implies (w, s) \in \bar{\Sigma}$ for all time.

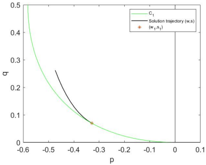
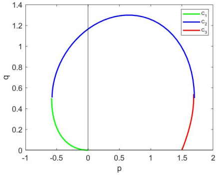


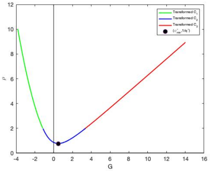
Figure: This cannot happen.

Compare the trajectory functions $s = \Gamma_1(p)$ and $\hat{q} = \Gamma_2(p)$.

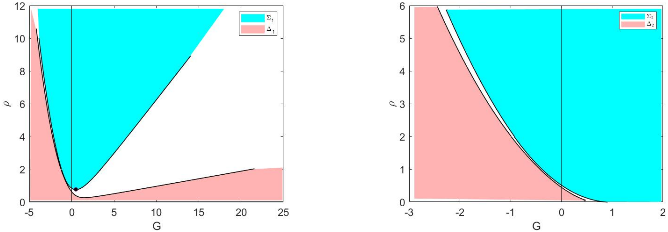
Retransformation to (G, ρ)



$$
\begin{aligned}
F(p,q) &= (p/q, 1/q) \\
&\approx (w/s, 1/s) \\
&= \left(\frac{G/\rho}{1/\rho}, \frac{1}{1/\rho}\right) \\
&\longrightarrow
\end{aligned}
$$



Global Solution (Subcritical) and FTB (Supercritical) Regions



(c) Weak alignment, $2\sqrt{\frac{k}{\bar{\rho}}} > \max \psi$ (d) Strong alignment, $2\sqrt{\frac{k}{\bar{\rho}}} \leq \min \psi$

Figure: Shapes of subcritical and supercritical regions.

References

- Critical thresholds in the Euler-Poisson-alignment system,
  MB, Hailiang Liu, Changhui Tan,
  ArXiv: 2111.11999.

THANK YOU