Critical thresholds in a nonlocal Euler system with relaxation

Manas Bhatnagar

CMS Seminar
University of Nebraska, Lincoln

Feb 25, 2021

This is joint work with my advisor, Dr Hailiang Liu.

This presentation is based on our papers submitted to *Discrete and Continuous Dynamical Systems:*

- Critical thresholds in a nonlocal Euler system with relaxation (ArXiv: 2010.02362),
- Sharp critical thresholds in a hyperbolic system with relaxation (ArXiv: 2012.06804).

Motivation to CT Phenomena

Pressureless Euler equation,

$$\rho_t + (\rho u)_x = 0, \quad \mathbb{R} \times (0, \infty),$$
$$u_t + u u_x = 0,$$

with $u(0, x) = u_0(x)$ and $\rho(0, x) = \rho_0(x) \geq 0$, both being C^1 functions.

Motivation to CT Phenomena

Pressureless Euler equation,

$$\begin{aligned} &\rho_t + (\rho u)_x = 0, \quad \mathbb{R} \times (0, \infty), \\ &u_t + u u_x = 0, \end{aligned}$$

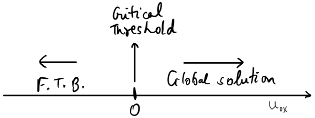
with $u(0,x) = u_0(x)$ and $\rho(0,x) = \rho_0(x) \geq 0$, both being C^1 functions. Using method of characteristics,

$$u_x(t, X(t;\alpha)) = \frac{u_{0x}(\alpha)}{1 + t u_{0x}(\alpha)}, \quad \alpha \in \mathbb{R}.$$

Hence, if for some x^*, $u_{0x}(x^*) < 0$, then

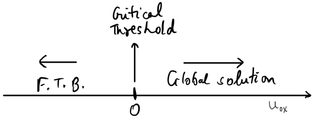
$$\lim_{t \to t^{*-}} u_{0x} = -\infty, \quad t^* = -\frac{1}{u_{0x}(x^*)}.$$

u_x is bounded for all time if and only if $u_{0x}(x) \geq 0$ for all $x \in \mathbb{R}$.



Criteria of breakdown: $u_x \to -\infty$ (shock formation) along with $\rho \to \infty$ (density concentration) at the same time.

u_x is bounded for all time if and only if $u_{0x}(x) \geq 0$ for all $x \in \mathbb{R}$.



Criteria of breakdown: $u_x \to -\infty$ (shock formation) along with $\rho \to \infty$ (density concentration) at the same time.
Following are some of the many researchers having extensively studied CTC for various systems:

- Hailiang Liu
  Eitan Tadmor
  Alexander Kiselev
  ...

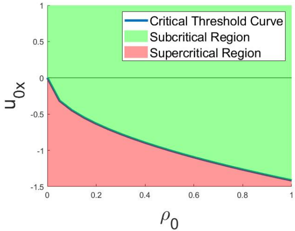
1D Euler-Poisson systems

$$
\begin{aligned}
&\rho_t + (\rho u)_x = 0, \quad x \in \mathbb{R} \\
&u_t + u u_x = -\phi_x, \\
&-\phi_{xx} = \rho.
\end{aligned}
$$

1D Euler-Poisson systems

$$
\begin{aligned}
&\rho_t + (\rho u)_x = 0, \quad x \in \mathbb{R} \\
&u_t + u u_x = -\phi_x, \\
&-\phi_{xx} = \rho.
\end{aligned}
$$

- There is global solution for all time if and only if $u_{0x}(x) + \sqrt{2\rho_0(x)} \geq 0$ for all $x \in \mathbb{R}$.



Model

A class of 1D Euler type system:

$$\rho_t + (\rho v)_x = 0, \quad x \in \mathbb{R}$$
$$u_t + u u_x = \rho(v - u).$$

with initial conditions $0 \leq \rho_0, u_0 \in L^\infty(\mathbb{R})$ and $\rho_x, u_x \in H^s(\mathbb{R})$, $s \geq 1$.

Model

A class of 1D Euler type system:

$$\rho_t + (\rho v)_x = 0, \quad x \in \mathbb{R}$$
$$u_t + u u_x = \rho(v - u).$$

with initial conditions $0 \le \rho_0, u_0 \in L^\infty(\mathbb{R})$ and $\rho_x, u_x \in H^s(\mathbb{R})$, $s \ge 1$.

- Choosing an appropriate v to close the system
- A nonlocal v has analogy with nonlocal models for fluid flows

Model

A class of 1D Euler type system:

$$\rho_t + (\rho v)_x = 0, \quad x \in \mathbb{R}$$
$$u_t + u u_x = \rho(v - u).$$

with initial conditions $0 \le \rho_0, u_0 \in L^\infty(\mathbb{R})$ and $\rho_x, u_x \in H^s(\mathbb{R})$, $s \ge 1$.

- Choosing an appropriate v to close the system
- A nonlocal v has analogy with nonlocal models for fluid flows
- We assume $v = Q * u$.

$v = Q * u$

$$\rho_t + \nabla \cdot (\rho Q * \mathrm{u}) = 0,$$
$$\mathrm{u}_t + (\mathrm{u} \cdot \nabla)\mathrm{u} = \rho(Q * \mathrm{u} - \mathrm{u}).$$

- $\int_{\mathbb{R}^N} Q(y)dy = 1$, $Q \geq 0$ and $Q \in W^{1,1}(\mathbb{R}^N)$
- For general Q, this system tends to a strictly hyperbolic system under the hyperbolic scaling limit $x \to x/\epsilon$, $t \to t/\epsilon$
- Assume the usual symmetricity of Q around the origin
- Conventional local existence uniqueness theorems for $\rho, \mathrm{u} \in H^{s+1}(\mathbb{R}^N)$, $s > N/2$
- We prove for more general space: $\rho, \mathrm{u} \in L^\infty(\mathbb{R}^N)$ and $\nabla\rho, \nabla\mathrm{u} \in H^s(\mathbb{R}^N)$

Local existence

- Follow a procedure inspired by Majda (1984)

Local existence

- Follow a procedure inspired by Majda (1984)

1. Set up an iteration scheme
2. Prove sequence is Cauchy in L^∞ norm (New)
3. Higher regularity of limit
4. Uniqueness using modified energy estimate derived

CTC in 1D

Question: Find a curve in the phase space of initial configuration which precisely delineates the global solution/finite-time-breakdown regions.

CTC in 1D

Question: Find a curve in the phase space of initial configuration which precisely delineates the global solution/finite-time-breakdown regions.

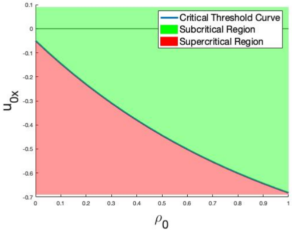


Figure: Idea of threshold curve.

Global solution: If $g(\rho_0(x), u_{0x}(x)) \geq 0$ for all $x \in \mathbb{R}$,
FTB: If $g(\rho_0(x^*), u_{0x}(x^*)) < 0$ for some $x^* \in \mathbb{R}$.

Result

$$\rho_t + (\rho Q * u)_x = 0, \quad x \in \mathbb{R}$$
$$u_t + uu_x = \rho(Q * u - u).$$

Theorem

- [Subcritical region] A unique solution $\rho, u \in C([0,\infty); L^\infty(\mathbb{R}))$ and
$$\rho_x, u_x \in C([0,\infty); H^s(\mathbb{R})), \quad s \geq 1$$
exists if $u_{0x}(x) + \rho_0(x) \geq 0$ for all $x \in \mathbb{R}$.
- [Supercritical region] If $\exists x^* \in \mathbb{R}$ for which $u_{0x}(x^*) < -\rho_0(x^*)$, then $u_x \to -\infty$ in a finite time.

Result

$$\rho_t + (\rho Q * u)_x = 0, \quad x \in \mathbb{R}$$
$$u_t + uu_x = \rho(Q * u - u).$$

Theorem

- [Subcritical region] A unique solution $\rho, u \in C([0,\infty); L^\infty(\mathbb{R}))$ and
$$\rho_x, u_x \in C([0,\infty); H^s(\mathbb{R})), \quad s \geq 1$$
exists if $u_{0x}(x) + \rho_0(x) \geq 0$ for all $x \in \mathbb{R}$.
- [Supercritical region] If $\exists x^* \in \mathbb{R}$ for which $u_{0x}(x^*) < -\rho_0(x^*)$, then $u_x \to -\infty$ in a finite time.

- Usually, it's hard to derive precise thresholds for nonlocal systems
- Remarkably simple for a nonlocal system

Local v

$$\rho_t + (\rho f(\rho, u))_x = 0,$$
$$u_t + u u_x = \rho(f(\rho, u) - u).$$

Question: For what class of f does this system exhibit critical threshold phenomena?

Local v

$$
\begin{aligned}
\rho_t + (\rho f(\rho, u))_x &= 0, \\
u_t + u u_x &= \rho(f(\rho, u) - u).
\end{aligned}
$$

Question: For what class of f does this system exhibit critical threshold phenomena?
Rewrite

$$
\begin{bmatrix} \rho \\ u \end{bmatrix}_t + \begin{bmatrix} \rho f_\rho + f & \rho f_u \\ 0 & u \end{bmatrix} \begin{bmatrix} \rho \\ u \end{bmatrix}_x = \begin{bmatrix} 0 \\ \rho(f - u) \end{bmatrix}.
$$

$\lambda_1 = \rho f_\rho + f$ and $\lambda_2 = u$.

$$\begin{bmatrix} \rho \\ u \end{bmatrix}_t + \begin{bmatrix} \rho f_\rho + f & \rho f_u \\ 0 & u \end{bmatrix} \begin{bmatrix} \rho \\ u \end{bmatrix}_x = \begin{bmatrix} 0 \\ \rho(f-u) \end{bmatrix}.$$

- $\lambda_1(\rho_0(x), u_0(x)) \neq u_0(x) \not\Rightarrow \lambda_1(\rho(t,x), u(t,x)) \neq u(t,x)$
- May change type: strictly hyperbolic to weakly hyperbolic

$$\begin{bmatrix} \rho \\ u \end{bmatrix}_t + \begin{bmatrix} \rho f_\rho + f & \rho f_u \\ 0 & u \end{bmatrix} \begin{bmatrix} \rho \\ u \end{bmatrix}_x = \begin{bmatrix} 0 \\ \rho(f-u) \end{bmatrix}.$$

- $\lambda_1(\rho_0(x), u_0(x)) \neq u_0(x) \not\Rightarrow \lambda_1(\rho(t,x), u(t,x)) \neq u(t,x)$
- May change type: strictly hyperbolic to weakly hyperbolic
- Divide into two cases owing to type of system:
  1. $f_\rho = 0$ ($f = f(u)$ only)
  2. General f

$f_\rho = 0$

- $\lambda_1 = f$
- $f - u > 0$ is invariant:
  $|f(u_0(x)) - u_0(x)| > 0 \implies |f(u(t,x)) - u(t,x)| > 0$

$f_\rho = 0$

- $\lambda_1 = f$
- $f - u > 0$ is invariant:
  $|f(u_0(x)) - u_0(x)| > 0 \implies |f(u(t,x)) - u(t,x)| > 0$

Proposition

Let f be a smooth function with $f_\rho = 0$. Consider initial conditions $(\rho_0 \geq 0, u_0) \in C^1_b(\mathbb{R}) \times C^1_b(\mathbb{R})$ with $\inf |f(u_0) - u_0| > 0$. If $f_u \leq 0$ for solution u under consideration, then

Bounds on u and ρ: $u(t,\cdot)$ is uniformly bounded and satisfies $|f(u(t,\cdot)) - u(t,\cdot)| > 0$ for $t > 0$. And

$$\rho(t,x) \leq \frac{\sup \rho_0 |f(u_0) - u_0|}{e^{\int_{u_0(x)}^{u(t,x)} \frac{d\xi}{f(\xi)-\xi}} |f(u(t,x)) - u(t,x)|}.$$

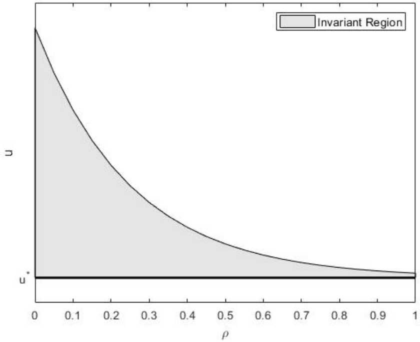


Figure: Asymptotic invariant region.

u^* is a root of $f(u) - u = 0$.

Theorem

Let f be a smooth function with $f_\rho = 0$. Consider initial conditions $(\rho_0 \geq 0, u_0) \in C_b^1(\mathbb{R}) \times C_b^1(\mathbb{R})$ with $\inf |f(u_0) - u_0| > 0$. If $f_u \leq 0$ for solution u under consideration, then

1. **Global solution:** If
$$u_{0x}(x) + \rho_0(x) \geq 0, \quad \forall x \in \mathbb{R},$$
then there exists a global classical solution $\rho, u \in C^1((0, \infty) \times \mathbb{R})$. Moreover, ρ, u_x are uniformly bounded.
2. **Finite time breakdown:** If $\exists x_0 \in \mathbb{R}$ such that
$$u_{0x}(x_0) + \rho_0(x_0) < 0,$$
then $\lim_{t \to t_c} |\rho_x| = \infty$ or $\lim_{t \to t_c} |u_x| = \infty$ for some $t_c > 0$.

Important points

- System is strictly hyperbolic
- Both ρ, u exist in the same space
- This is consistent with the usual results for a strictly hyperbolic system

Idea of proof

Take derivative of $u_t + uu_x = \rho(f - u)$ and set $e := u_x + \rho$,

$$\rho_t + f\rho_x = f_u\rho(\rho - e),$$
$$e_t + ue_x = -e(e - \rho).$$

Idea of proof

Take derivative of $u_t + uu_x = \rho(f - u)$ and set $e := u_x + \rho$,

$$\rho_t + f\rho_x = f_u\rho(\rho - e),$$
$$e_t + ue_x = -e(e - \rho).$$

- If $f_u \leq 0$, $e_0 \geq 0$, we can show $\sup e_0(x), \sup \rho_0(x) \leq M \implies 0 \leq e(t,x), \rho(t,x) \leq M$ for all $x \in \mathbb{R}$ and $t > 0$.
- By taking derivative of first equation, ρ_x is bounded for any time if e is.
- For uniform bound on ρ, we need not assume $\inf |f(u_0) - u_0| > 0$. However, we need to use u_x to bound.

- By assuming strict hyperbolicity, we can use Riemann Invariant technique (P.D. Lax, 1964).

$$\begin{bmatrix} \rho \\ u \end{bmatrix}_t + \begin{bmatrix} f & \rho f_u \\ 0 & u \end{bmatrix} \begin{bmatrix} \rho \\ u \end{bmatrix}_x = \begin{bmatrix} 0 \\ \rho(f-u) \end{bmatrix}$$

$$\phi[f-u \quad \rho f_u] \left(\begin{bmatrix} \rho \\ u \end{bmatrix}_t + \begin{bmatrix} f & \rho f_u \\ 0 & u \end{bmatrix} \begin{bmatrix} \rho \\ u \end{bmatrix}_x = \begin{bmatrix} 0 \\ \rho(f-u) \end{bmatrix} \right).$$

If $\exists R = R(\rho, u)$ s.t. $[R_\rho \quad R_u] = \phi[f-u \quad \rho f_u]$,

- By assuming strict hyperbolicity, we can use Riemann Invariant technique (P.D. Lax, 1964).

$$\begin{bmatrix} \rho \\ u \end{bmatrix}_t + \begin{bmatrix} f & \rho f_u \\ 0 & u \end{bmatrix} \begin{bmatrix} \rho \\ u \end{bmatrix}_x = \begin{bmatrix} 0 \\ \rho(f-u) \end{bmatrix}$$
$$\phi[f-u \quad \rho f_u] \left(\begin{bmatrix} \rho \\ u \end{bmatrix}_t + \begin{bmatrix} f & \rho f_u \\ 0 & u \end{bmatrix} \begin{bmatrix} \rho \\ u \end{bmatrix}_x = \begin{bmatrix} 0 \\ \rho(f-u) \end{bmatrix} \right).$$

If $\exists R = R(\rho, u)$ s.t. $[R_\rho \quad R_u] = \phi[f-u \quad \rho f_u]$,

$$[R_\rho \quad R_u] \left(\begin{bmatrix} \rho \\ u \end{bmatrix}_t + \begin{bmatrix} f & \rho f_u \\ 0 & u \end{bmatrix} \begin{bmatrix} \rho \\ u \end{bmatrix}_x = \begin{bmatrix} 0 \\ \rho(f-u) \end{bmatrix} \right)$$
$$R_\rho \rho_t + R_u u_t + f(R_\rho \rho_x + R_u u_x) = R_u \rho(f-u)$$
$$R_t + fR_x = \phi \rho^2 f_u (f-u) = \rho R f_u.$$

- $R = \rho e^{\int_{u_0}^{u} \frac{d\xi}{f(\xi)-\xi}}(f-u) = \rho\phi(f-u)$. Asymptotic bounds on ρ can be readily concluded.

- We do **not** need u_x.

- We do **not** need u_x.

Differentiating equation of R

$$(R_x)_t + fR_{xx} = -f_u r(e-3\rho) - \rho R(e-\rho)\left(\frac{f_u^2}{f-u} - f_{uu}\right).$$

- Boundedness of R_x and, in turn, ρ_x can be readily concluded
- ρ and u bounded together AND ρ_x and u_x together.

General f

Theorem

Let $f = f(\rho, u)$ be a smooth function of its variables. Consider initial conditions $(\rho_0 \geq 0, u_0) \in C_b^1(\mathbb{R}) \times C_b^2(\mathbb{R})$. If $f_u \leq 0$ for the solutions under consideration, then

1. **Bounds on u:** There exists a smooth function $\phi : \mathbb{R}^+ \to \mathbb{R}$ such that,

$$\min\left\{\inf_{\mathbb{R}} u_0, \min \phi(\rho)\right\} \leq u(t, \cdot) \leq \max\left\{\sup_{\mathbb{R}} u_0, \max \phi(\rho)\right\},$$

   for as long as $\rho \geq 0$ is bounded.

2. **Bounds on ρ, u_x:** If

$$u_{0x}(x) + \rho_0(x) \geq 0, \quad \forall x \in \mathbb{R},$$

   then ρ, u_x are uniformly bounded.

Idea of proof

$$u_t + uu_x = \rho(f(\rho, u) - u)$$

- $f_u - 1 < 0$, then an Implicit Function Theorem type result gives the existence of $\phi = \phi(\rho)$ with $f(\rho, \phi(\rho)) \equiv \phi(\rho)$.
- Owing to the structure, we can show

$$\min\{\inf u_0, \min \phi(\rho)\} \leq u(t, \cdot) \leq \max\{\sup u_0, \max \phi(\rho)\}.$$

- Bounds on u_x is similar to previous case (Using the quantity $e = u_x + \rho$).

Global Solution

Theorem

Let $f = f(\rho, u)$ be a smooth function of its variables. Consider initial conditions $(\rho_0 \geq 0, u_0) \in C_b^1(\mathbb{R}) \times C_b^2(\mathbb{R})$. If $f_u \leq 0$ for the solutions under consideration, then

Global solution: If in addition to $u_{0x} + \rho_0 \geq 0$,

- $(\rho f)_{\rho\rho} \geq 0,\ f_{uu} \leq 0$ along with

$$\rho_{0x}(x) \geq 0, \quad u_{0xx}(x) + \rho_{0x}(x) \geq 0,\ \forall x \in \mathbb{R},$$

OR

- $(\rho f)_{\rho\rho} \leq 0,\ f_{uu} \geq 0$ along with

$$\rho_{0x}(x) \leq 0, \quad u_{0xx}(x) + \rho_{0x}(x) \leq 0,\ \forall x \in \mathbb{R},$$

then there exists a global solution $\rho \in C^1((0, \infty) \times \mathbb{R})$ and $u \in C^2((0, \infty) \times \mathbb{R})$.

Important points

- More conditions needed on f
- u_0 and ρ_0 lie in different space: one more derivative need for u
- Same for ρ as well as u

Idea of proof

- $e = u_x + \rho$ can be bound as before
- $\xi = \rho_x$ and $\eta = e_x$:

$$\xi_t + \lambda_1 \xi_x = -(\rho f)_{\rho\rho}\xi^2 + \Big(2\rho f_{\rho u}(\rho - e) + f_u(3\rho - 2e)\Big)\xi - \rho f_u \eta - f_{uu}\rho(\rho - e)^2,$$

$$\eta_t + u\eta_x = e\xi + (2\rho - 3e)\eta.$$

Idea of proof

- $e = u_x + \rho$ can be bound as before
- $\xi = \rho_x$ and $\eta = e_x$:

$$\begin{aligned}\xi_t + \lambda_1 \xi_x &= -(\rho f)_{\rho\rho}\xi^2 + \Big(2\rho f_{\rho u}(\rho - e) + f_u(3\rho - 2e)\Big)\xi \\ &\quad - \rho f_u \eta - f_{uu}\rho(\rho - e)^2, \\ \eta_t + u\eta_x &= e\xi + (2\rho - 3e)\eta.\end{aligned}$$

- η is bounded if ξ is.

$$\frac{d\eta}{dt} = e\xi + (2\rho - 3e)\eta$$

gives

$$\eta(t, X) = \eta(0, x_0) exp^{\int (2\rho - 3e)\, d\tau} + \int_0^t exp^{\int_s^t (2\rho - 3e)\, d\tau} e\xi \, ds.$$

$$\eta(t,X) = \eta(0,x_0)exp^{\int (2\rho-3e)\,d\tau} + \int_0^t exp^{\int_s^t (2\rho-3e)\,d\tau} e\xi \, ds.$$

Lemma

Suppose $e \geq 0$. As long as ξ exists:

- If $\eta(0,x) \geq 0$ and $\xi(t,x) \geq 0$ for $t \geq 0, x \in \mathbb{R}$, then $\eta(t,x) \geq 0$ for $t \geq 0, x \in \mathbb{R}$,
- If $\eta(0,x) \leq 0$ and $\xi(t,x) \leq 0$ for $t \geq 0, x \in \mathbb{R}$, then $\eta(t,x) \leq 0$ for $t \geq 0, x \in \mathbb{R}$.

Using this, we can derive the final Proposition.

Proposition

(Bounds on ξ) Assume $e(0, x) \geq 0 \ \forall x \in \mathbb{R}$.

- Suppose $\eta(0, x), \xi(0, x) \geq 0$ for all $x \in \mathbb{R}$. Also assume $(\rho f)_{\rho\rho} \geq 0$ and $f_{uu} \leq 0$ for solutions under consideration. Then $\eta(t, x) \geq 0$, $\xi(t, x) \geq 0$, for $t > 0, x \in \mathbb{R}$ and for any $t > 0$, $\xi(t, \cdot)$ has a uniform upper bound.
- Suppose $\eta(0, x), \xi(0, x) \leq 0$ for all $x \in \mathbb{R}$. Also assume $(\rho f)_{\rho\rho} \leq 0$ and $f_{uu} \geq 0$ for solutions under consideration. Then $\eta(t, x) \leq 0$, $\xi(t, x) \leq 0$, for $t > 0, x \in \mathbb{R}$ and for any $t > 0$, $\xi(t, \cdot)$ has a uniform lower bound.

Finite time breakdown

Theorem

Let $f = f(\rho, u)$ be a smooth function of its variables. Consider initial conditions $(\rho_0 \geq 0, u_0) \in C_b^1(\mathbb{R}) \times C_b^2(\mathbb{R})$. If $f_u \leq 0$ for the solutions under consideration, then

Finite time breakdown: If $\exists x_0 \in \mathbb{R}$ such that

$$u_{0x}(x_0) + \rho_0(x_0) < 0,$$

then $\lim_{t \to t_c^-} u_x = -\infty$ at the rate of $O\left(\frac{1}{|t - t_c|}\right)$ or faster, for some $t_c > 0$.

Summary

- For nonlocal relaxation,
  1. Local existence in a more general space,
  2. Precise critical threshold.
- For local relaxation,
  1. Existence of precise threshold in a special case under
  2. Bounds on both thresholds under some restrictions, hence the existence of critical threshold
  3. Proof of local existence by modified existing proof for strictly hyperbolic systems (C.M. Dafermos)

References

- S. Engelberg, H. Liu, E. Tadmor,
  Critical thresholds in Euler-Poisson equations,
  Indiana University Math. Journal, 50:109–157, 2001.
- C.M. Dafermos,
  Hyperbolic Conservation Laws in Continuum Physics,
  Springer-Verlag Berlin Heidelberg, Vol. 325, 2010 (3rd Ed.).
- P. Lax.,
  Development of singularities of solutions of nonlinear hyperbolic partial differential equations,
  Journal of Math. Phys., 5, 611, 1964.
- H. Liu, E. Tadmor,
  Critical thresholds in a convolution model for nonlinear conservation laws,
  SIAM J. Math. Anal., 33(4): 930–945, 2001.
- A. Majda,
  Compressible Fluid Flow and Systems of Conservation Laws in Several Space Variables,
  Springer Science+Business Media, Vol. 53, 1984.
- G. R. Baker, X. Li, and A. C. Morlet,
  Analytic structure of two 1D-transport equations with fluxes,
  Physica D, 91: 349–375, 1996.

References

- M. Bhatnagar, H. Liu,
  Critical thresholds in one-dimensional damped Euler-Poisson systems,
  Math. Mod. Meth. Appl. Sci., 30(5):891–916, 2020.
- M. Bhatnagar, H. Liu,
  Critical thresholds in 1D pressureless Euler-Poisson systems with variable background,
  Physica D: Nonlinear Phenomena, 414, 2020.
- T. Li, H. Liu,
  Critical Thresholds in a relaxation system with resonance of characteristic speeds,
  Disc. Cont. Dyn. Sys-Series A, 24(2):511–521, 2009.
- T. Li, H. Liu,
  Critical thresholds in hyperbolic relaxation systems,
  J. Diff. Eqns., 247:33–48, 2009.

A natural question

- Is there an existence of a critical threshold for $f_u > 0$?

A natural question

- Is there an existence of a critical threshold for $f_u > 0$?

Thank You!

Questions?