# Critical Thresholds in a nolocal Euler system


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Abstract

We propose and study a nonlocal Euler system with relaxation, which tends to a strictly hyperbolic system under the hyperbolic scaling limit. We derive a precise critical threshold for this system in one dimensional setting. Our result reveals that such nonlocal system admits global smooth solutions for a large class of initial data. Thus, the nonlocal velocity regularizes the generic finite-time breakdown in the pressureless Euler system. We also look into a one-dimensional $2 \times 2$ hyperbolic Eulerian system with local relaxation from critical threshold perspective. The system features dynamic transition between strictly and weakly hyperbolic.


## Introduction

We are concerned with the critical threshold phenomena in the following Euler system,

$$\rho_t + (\rho v)_x = 0, \quad x \in \mathbb{R} \tag{1a}$$
$$u_t + u u_x = \rho(v - u), \tag{1b}$$

with initial data $\rho_0 \geqslant 0, u_0$. We need to choose an appropriate $v$ to close the system. A nonlocal $v$ has analogy with some fluid flow models. We consider two cases: (1) $v = Q * u$, (2) $v = f(\rho, u)$. $Q : \mathbb{R} \to \mathbb{R}^+$ is the interaction kernel with $Q \in W^{1,1}(\mathbb{R})$ and unit weight. Therefore, $Q * u$ is kind of a weighted average of $u$. Assuming this we can show that under a hyperbolic scaling ($x \to x/\epsilon$, $t \to t/\epsilon$), (1) converges to a strictly hyperbolic system. This asymptotic analysis holds in multidimension as well and the limiting system is,

$$\rho_t + \nabla \cdot (\rho \mathbf{u}) = 0,$$
$$\mathbf{u}_t + ((\mathbf{u} - \rho\gamma) \cdot \nabla)\mathbf{u} = 0, \qquad \gamma = \int_{\mathbb{R}^N} y Q(y) dy.$$

However, if $v = f(\rho, u)$ is local then,

$$\begin{bmatrix} \rho \\ u \end{bmatrix}_t + \begin{bmatrix} \rho f_\rho + f & \rho f_u \\ 0 & u \end{bmatrix} \begin{bmatrix} \rho \\ u \end{bmatrix}_x = \begin{bmatrix} 0 \\ \rho(f - u) \end{bmatrix},$$

is a mixed system that could change from strictly hyperbolic to weakly hyperbolic at a later time because the difference of the two eigenvalues, $\lambda_1 - \lambda_2 = \rho f_\rho(\rho, u) + f - u$, could be zero for some later time, if not initially. This unique feature reflects heavily upon the analysis as well as the results.

## Key Idea

Taking derivative of (1b) and using (1a), we obtain a transport equation for $e := u_x + \rho$. Coupled with (1a), we obtain,

$$\rho_t + \lambda_1 \rho_x = f_u \rho(\rho - e),$$
$$e_t + u e_x = -e(e - \rho).$$

Given $e_0 \geqslant 0$, the above system allows us to show $e, \rho \in [0, M]$, $M = \max\{\sup e_0, \sup \rho_0\}$. If system is strictly hyperbolic, then Riemann invariant technique [1] can be leveraged. Indeed, when $f_\rho = 0$, $\lambda_1 - \lambda_2 = f - u$ maintains sign for all time making the system strictly hyperbolic.

# Results

## Theorem (Local existence for nonlocal system [3])

*Consider $v = Q * u$ in (1) and spatial dimension being $N \geqslant 1$. Let $s > N/2$ be a positive integer. Suppose $\rho_0 \in L^\infty(\mathbb{R}^N)$, $\mathbf{u_0} \in (L^\infty(\mathbb{R}^N))^N$ and $D^1\rho_0 \in H^s(\mathbb{R}^N)$, $D^1\mathbf{u_0} \in (H^s(\mathbb{R}^N))^N$. Then for any $L > 0$, if*

$$\max\{\|\rho_0\|_\infty, \|\nabla\rho_0\|_{H^s(\mathbb{R}^N)}, \|\mathbf{u_0}\|_\infty, \|\nabla\mathbf{u_0}\|_{H^s(\mathbb{R}^N)}\} \leqslant L,$$

*then $\exists T > 0$, depending on $L$, $Q$ and continuously differentiable functions $\rho$, $\mathbf{u}$ satisfying*

$$\max\left\{\sup_{[0,T]} \|\rho(t,\cdot)\|_\infty, \sup_{[0,T]} \|\nabla\rho(t,\cdot)\|_{H^s(\mathbb{R}^N)}, \sup_{[0,T]} \|\mathbf{u}(t,\cdot)\|_\infty, \|\nabla\mathbf{u}(t,\cdot)\|_{H^s(\mathbb{R}^N)}\right\} \leqslant 2L,$$

*which are classical solutions.*

Note that this result allows for initial data that need not decay at infinity, which is in contrast to the conventional existence-uniqueness theorems for conservation laws using energy method.

## Theorem (Sharp threshold for nonlocal system [3])

*Consider $v = Q * u$ in (1).*

- *[Subcritical region] A unique solution $\rho, u \in C([0,\infty); L^\infty(\mathbb{R}))$ and*
  $$\rho_x, u_x \in C([0,\infty); H^s(\mathbb{R})), \quad s \geqslant 1$$
  *exists if $u_{0x}(x) + \rho_0(x) \geqslant 0$ for all $x \in \mathbb{R}$.*
- *[Supercritical region] If $\exists x^* \in \mathbb{R}$ for which $u_{0x}(x^*) < -\rho_0(x^*)$, then $u_x \to -\infty$ in a finite time.*

## Theorem (Sharp threshold for special case of local system [4])

*Let $f$ be a smooth function depending on $u$ only, i.e., $f_\rho = 0$. Consider the system (1) with $v = f(u)$ and initial conditions $\rho_0 \geqslant 0$, $u_0 \in C_b^1(\mathbb{R})$. Suppose $\inf|f(u_0) - u_0| > 0$. If $f_u \leqslant 0$ for solution $u$ under consideration, then*

1. ***Bounds on $u$ and $\rho$:** $u(t,\cdot)$ is uniformly bounded and satisfies $|f(u(t,\cdot)) - u(t,\cdot)| > 0$ for $t > 0$. And*
   $$\rho(t,x) \leqslant \frac{\sup \rho_0 |f(u_0) - u_0|}{e^{\int_{u_0(x)}^{u(t,x)} \frac{d\xi}{f(\xi)-\xi}} |f(u(t,x)) - u(t,x)|}.$$
2. ***Global solution:** If*
   $$u_{0x}(x) + \rho_0(x) \geqslant 0, \quad \forall x \in \mathbb{R},$$
   *then there exists a global classical solution $\rho, u \in C^1((0,\infty) \times \mathbb{R})$ to (1). Moreover, $\rho, u_x$ are uniformly bounded with*
   $$0 \leqslant \rho(t,x) \leqslant M, \quad 0 \leqslant u_x(t,x) + \rho(t,x) \leqslant M, \qquad \forall t > 0, x \in \mathbb{R},$$
   *where $M = \max\{\sup\rho_0, \sup(u_{0x} + \rho_0)\}$.*
3. ***Finite time breakdown:** If $\exists x_0 \in \mathbb{R}$ such that*
   $$u_{0x}(x_0) + \rho_0(x_0) < 0,$$
   *then $\lim_{t\to t_c} |\rho_x| = \infty$ or $\lim_{t\to t_c} |u_x| = \infty$ for some $t_c > 0$.*

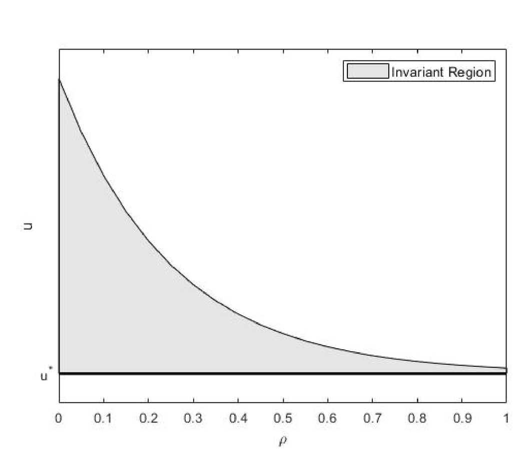


Figure: Asymptotic invariant region

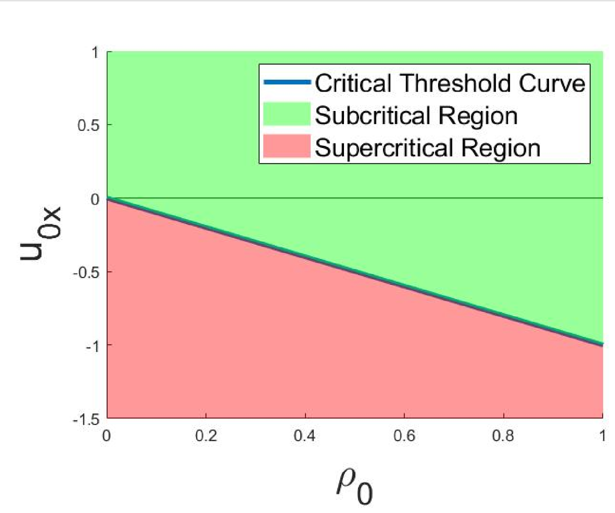


Figure: Critical threshold curve

## Theorem (Bounds on thresholds for general system [4])

*Let $f = f(\rho, u)$ be a smooth function of its variables. Consider the system (1) with $v = f(\rho, u)$ and initial conditions $(\rho_0 \geqslant 0, u_0) \in C_b^1(\mathbb{R}) \times C_b^2(\mathbb{R})$. If $f_u \leqslant 0$ for the solutions under consideration, then*

1. ***Bounds on $u$:** There exists a smooth function $\phi : \mathbb{R}^+ \to \mathbb{R}$ such that,*
   $$\min\left\{\inf_{\mathbb{R}} u_0, \min\phi(\rho)\right\} \leqslant u(t,\cdot) \leqslant \max\left\{\sup_{\mathbb{R}} u_0, \max\phi(\rho)\right\},$$
   *for as long as $\rho \geqslant 0$ is bounded.*
2. ***Bounds on $\rho, u_x$:** If*
   $$u_{0x}(x) + \rho_0(x) \geqslant 0, \quad \forall x \in \mathbb{R},$$
   *then $\rho, u_x$ are uniformly bounded with*
   $$0 \leqslant \rho(t,x) \leqslant M, \quad 0 \leqslant u_x(t,x) + \rho(t,x) \leqslant M, \qquad \forall t > 0, x \in \mathbb{R}.$$
   *where $M = \max\{\sup\rho_0, \sup(u_{0x} + \rho_0)\}$.*
3. ***Global solution:** If in addition to $u_{0x} + \rho_0 \geqslant 0$,*
   - *$(\rho f)_{\rho\rho} \geqslant 0$, $f_{uu} \leqslant 0$ along with*
     $$\rho_{0x}(x) \geqslant 0, \quad u_{0xx}(x) + \rho_{0x}(x) \geqslant 0, \ \forall x \in \mathbb{R},$$
     *OR*
   - *$(\rho f)_{\rho\rho} \leqslant 0$, $f_{uu} \geqslant 0$ along with*
     $$\rho_{0x}(x) \leqslant 0, \quad u_{0xx}(x) + \rho_{0x}(x) \leqslant 0, \ \forall x \in \mathbb{R},$$

   *then there exists global solution $\rho \in C^1((0,\infty) \times \mathbb{R})$, $u \in C^2((0,\infty) \times \mathbb{R})$.*
4. ***Finite time breakdown:** If $\exists x_0 \in \mathbb{R}$ such that*
   $$u_{0x}(x_0) + \rho_0(x_0) < 0,$$
   *then $\lim_{t\to t_c} u_x = -\infty$ at the rate of $O\left(\frac{1}{|t-t_c|}\right)$ or faster, for some $t_c > 0$.*

## Remark

If in the hypothesis of the above Theorem, $f_\rho = 0$, then a global solution exists **iff** $u_{0x}(x) + \rho_0(x) \geqslant 0$ for all $x \in \mathbb{R}$. In this situation, $\rho_x$ is bounded as a result of $u_x$, $\rho$ being bounded. However, given the hypothesis of the Theorem above, we need to bound $\rho_x$ separately which gives rise to the extra conditions needed in assertion (3).

## Summary

We obtain precise threshold for the nonlocal system ($v = Q * u$). Our analysis can be extended to a larger class of nonlocal models of similar type. For the local relaxation case ($v = f(\rho, u)$), we see that it exhibits critical threshold phenomena for a certain class of $f$. $f_\rho = 0$ is special case where system can be strictly hyperbolic for a class of initial data. The analysis to obtain bounds on $\rho$ is different depending on the type of system, weakly or strictly hyperbolic. Question of the existence of a threshold still stands when $f_u \nleq 0$. This is left for future study.

## References


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