Critical Threshold Phenomena in 1D pressureless Euler-Poisson systems with variable background

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This is joint work with my advisor, Dr Hailiang Liu.

This presentation is based on our paper, published in:
Physica D: Nonlinear Phenomena, (414), '20.
Precursor: Critical Thresholds in one dimensional damped Euler-Poisson systems, *Math. Mod. Meth. Appl. Sci.*, 30(5): 891-916, 2020.
Under review: Critical thresholds in a nonlocal Euler system with relaxation, *Submitted in Journal of Differential Equations*

Introduction

Motivation to CTC

Pressureless Euler equation,

$$\begin{aligned} &\rho_t + (\rho u)_x = 0, \quad \mathbb{R} \times (0, \infty), \\ &u_t + u u_x = 0, \end{aligned}$$

with $u(0, x) = u_0(x)$.

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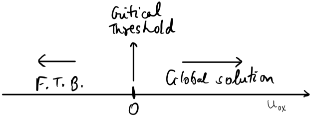
with $u(0, x) = u_0(x)$. Using method of characteristics,

$$u_x(t, X(t; \alpha)) = \frac{u_{0x}(\alpha)}{1 + t u_{0x}(\alpha)}, \quad \alpha \in \mathbb{R}.$$

Hence, if for some x^*, $u_{0x}(x^*) < 0$, then

$$\lim_{t \to t^{*-}} u_{0x} = -\infty, \quad t^* = -\frac{1}{u_{0x}(x^*)}.$$

u_x is bounded for all time if and only if $u_{0x}(x) \geq 0$ for all $x \in \mathbb{R}$.



Criteria of breakdown: $u_x \to -\infty$ (shock formation) along with $\rho \to \infty$ (density concentration) at the same time.

Model

A class of 1D Euler-Poisson equations with background and damping:

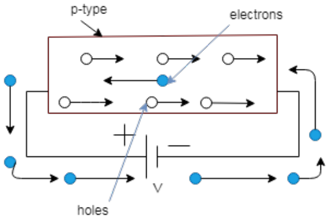
$$\rho_t + (\rho u)_x = 0, \quad x \in \mathbb{R} \quad \text{(Mass conservation)}$$
$$u_t + uu_x + \nu u = -k\phi_x, \quad \text{(Momentum equation)}$$
$$-\phi_{xx} = \rho - c, \quad \text{(Poisson equation)}$$

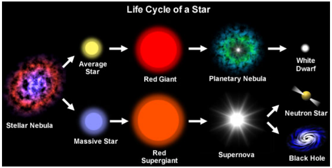
with initial conditions $(\rho_0(x) \geq 0, u_0(x)) \in C_b^1(\mathbb{R}) \times C_b^1(\mathbb{R})$.

- ν is the damping constant.
- c is the background term which may be a constant or vary with space.
- k signifies property of underlying forcing: repulsive ($k > 0$) or attractive ($k < 0$).

Applications

- Modeling charge flows in semiconductors, cosmological waves and evolution of stars.





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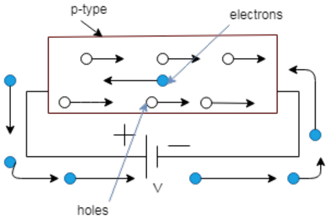

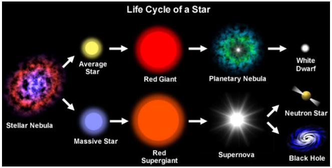

Following are some of the many researchers having extensively studied regularity of Euler-Poisson systems:

- Hailiang Liu
  Eitan Tadmor
  Alexander Kiselev
  . . .

Critical Threshold Phenomena

Question: Find a curve in the phase space of initial configuration which precisely delineates the global solution/finite-time-breakdown regions.

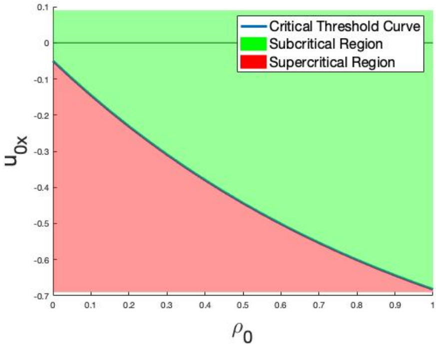

Figure: Idea of threshold curve in Euler Poisson systems.

Constant background

For $k > 0$ (repulsive forcing), there are 3 separate cases,

1. $\nu > 2\sqrt{kc}$, strong damping (No oscillations),
2. $\nu < 2\sqrt{kc}$, weak damping (Oscillatory solutions), and
3. $\nu = 2\sqrt{kc}$, borderline damping (No oscillations).

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3. $\nu = 2\sqrt{kc}$, borderline damping (No oscillations).

For $k < 0$ (attractive forcing),

- Solutions are of a single type without any oscillations.

Repulsive forcing and constant background

Theorem 1 (B-Liu '20, M3AS)

Let $k > 0$ and $\nu < 2\sqrt{kc}$ (weak damping). There exists a unique solution $\rho, u \in C^1(\mathbb{R} \times (0, \infty))$ iff $\forall x \in \mathbb{R}$,

$$(\rho_0(x), u_{0x}(x)) \in \{(\rho, d) : -\rho Q_1(1/\rho) < d < -\rho Q_2(s^* - 1/\rho),\ \rho \in (1/s^*, \infty)\},$$

where $s^* > 0$ is uniquely determined, and $Q_1 : [0, s^*] \longrightarrow \mathbb{R}^+ \cup \{0\}$ is a continuous function satisfying

$$\frac{dQ_1}{ds} = \nu + \frac{k}{Q_a}(1 - cs),\ \ Q_1(0) = 0,$$

and $Q_2 : [0, s^*] \longrightarrow \mathbb{R}^- \cup \{0\}$ is another continuous function satisfying

$$\frac{dQ_2}{ds} = -\nu + \frac{k}{Q_2}(c(s^* - s) - 1), \quad Q_2(0) = 0.$$

Repulsive forcing and constant background

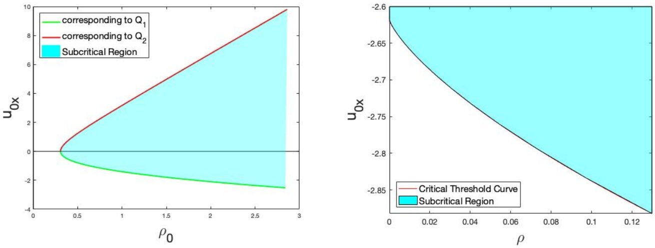


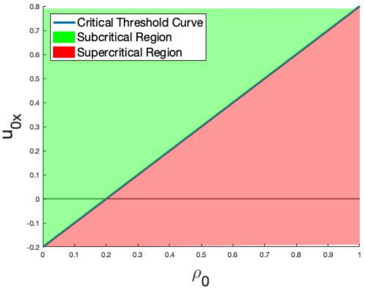
Figure: Left: Weak damping ($\nu < 2\sqrt{kc}$). Right: Strong damping ($\nu > 2\sqrt{kc}$).

Attractive forcing

Theorem 2

Let $k < 0$. Then there exists a unique solution $\rho, u \in C^1(\mathbb{R} \times (0,\infty))$ iff $\forall x \in \mathbb{R}$,

$$(\rho_0(x), u_{0x}(x)) \in \left\{ (\rho, d) : d \geq \left(\frac{\nu + \sqrt{\nu^2 - 4kc}}{2c} \right) (\rho - c) \right\}.$$



Variable background

- Question: What if $c = c(x)$?

$$\begin{aligned} \rho_t + (\rho u)_x &= 0, \\ u_t + u u_x + \nu u &= -k\phi_x, \\ -\phi_{xx} &= \rho - c(x), \end{aligned}$$

with $\rho_0 \geq 0$, $u_0 \in C^1(\mathbb{T})$.

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- Not just more general but physically more meaningful too.
- We will consider $k < 0$ and $c_1 \leq c(x) \leq c_2$.
- Criteria of breakdown: At time t^* of breakdown,

$$-\lim_{t \uparrow t^*} u_x(\cdot, t) = \lim_{t \uparrow t^*} \rho(\cdot, t) = \infty.$$

- Therefore, aim is to bound ρ from above and u_x from below.

Variable Background

Global solution

Theorem 3 (B-Liu '20, Phys D)

If for all $x \in \mathbb{T}$,

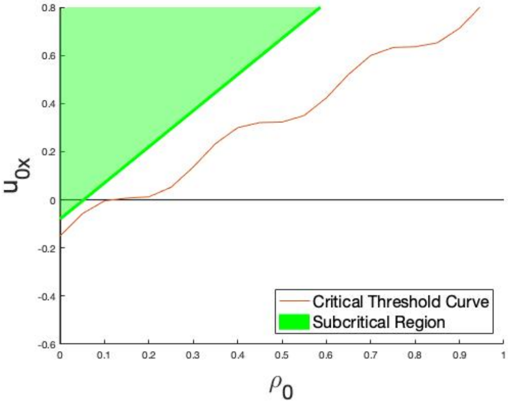
$$(\rho_0(x), u_{0x}(x)) \in \left\{(\rho, d) : d > \frac{\lambda_1}{c_1}(\rho - c_1)\right\},$$

then there exists a unique global-in-time solution, $\rho, u \in C^1((0,\infty) \times \mathbb{T})$. *Moreover,*

$$\begin{aligned} &||\rho(t,\cdot)||_\infty \leq ||\rho_0||_\infty e^{\lambda_1 t}, \\ &-\lambda_1 \leq u_x(t,\cdot) \leq constant(c_2, \nu, ||u_{0x}||_\infty). \end{aligned}$$

$$c_1 = \min c, \quad c_2 = \max c,$$

$$\lambda_1 = \frac{\nu + \sqrt{\nu^2 - 4kc_1}}{2}.$$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0.8
0.9
1
u_{0x}
ρ_0
Critical Threshold Curve
Subcritical Region

Finite-time-breakdown

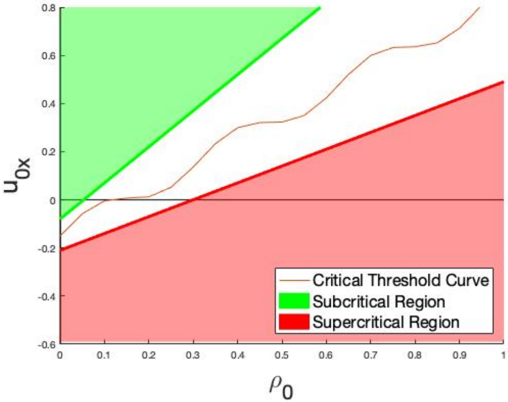
Theorem 4 (B-Liu '20, Phys D)

If $\exists x_0 \in \mathbb{T}$ *such that*

$$u_{0x}(x_0) < \left(\frac{\nu + \sqrt{\nu^2 - 4kc_2}}{2c_2}\right)(\rho_0(x_0) - c_2),$$

then $\exists(t^*, x^*)$ *such that*

$$\lim_{t\uparrow t^*} u_x(t, x^*) = -\lim_{t\uparrow t^*} \rho(t, x^*) = -\infty.$$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u0x
ρ0
Critical Threshold Curve
Subcritical Region
Supercritical Region

Preliminaries

Let $u_x := d$. Using method of characteristics, EP system can be reduced to ODE system:

$$\rho' = -\rho d,$$
$$d' = -d^2 - \nu d + k(\rho - c(x(t))),$$

Initial conditions:

$$\rho(0) = \rho_0(\alpha), \quad d(0) = d_0(\alpha) = u_{0x}(\alpha),$$

for $\alpha \in \mathbb{T}$.

Tools: Auxiliary system

- Introduce an auxiliary system of equations:

$$
\begin{aligned}
\eta' &= -\eta\xi, \\
\xi' &= -\xi^2 - \nu\xi + k\eta - k\gamma.
\end{aligned}
$$

$\gamma \geq 0$ is parameter. Initial conditions, $\eta(0), \xi(0)$, to be chosen appropriately later.

Proposition 2.1

If

$$\xi(0) \geq \left(\frac{\nu + \sqrt{\nu^2 - 4k\gamma}}{2\gamma}\right)(\eta(0) - \gamma),$$

then (η, ξ) *exist for all time and,*

$$\xi(t) \geq \left(\frac{\nu + \sqrt{\nu^2 - 4k\gamma}}{2\gamma}\right)(\eta(t) - \gamma).$$

Otherwise, $\exists t^*$ *such that*

$$\lim_{t \uparrow t^*} \eta(t) = -\lim_{t \uparrow t^*} \xi(t) = \infty.$$

Sketch of proof

$$\eta(t) = \eta(0)e^{-\int_0^t \xi d\tau}.$$

- As long as ξ exists, η maintains sign. Hence, $\eta(0) = 0 \equiv \eta(t)$ can be handled separately.
- For $\eta(0) > 0 \implies \eta(t) > 0$ for all time. An eventual transformation of variables:

$$r = \frac{\xi}{\eta}, \quad s = \frac{1}{\eta},$$

  enables us to derive the proposition.
- This crucial transformation has also been used by other authors (C. Tan, A. Kiselev,...) to analyze at CT for various class of Euler-Poisson systems.

Sketch of proof

$$\gamma s = 1 - \frac{r}{\lambda}, \quad \lambda = \frac{\nu + \sqrt{\nu^2 - 4k\gamma}}{2\gamma}.$$

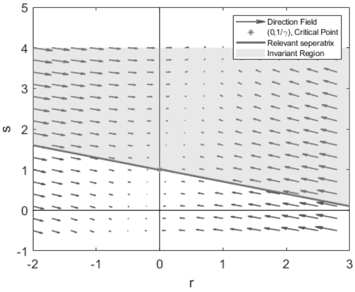

Figure: $r - s$ phase potrait.

Tools: Comparison Lemma

Lemma 5

As long as ρ, d, η, ξ exist, we have
(i) For $\gamma = \min c(x)$: If $\rho(0) < \eta(0)$ and $d(0) > \xi(0)$, then

$$\rho(t) < \eta(t), \quad d(t) > \xi(t).$$

(ii) For $\gamma = \max c(x)$: If $\rho(0) > \eta(0)$ and $d(0) < \xi(0)$, then

$$\rho(t) > \eta(t), \quad d(t) < \xi(t).$$

Comparison Lemma

- Proved by a contradiction argument.
- Sign of k plays a vital role.
- A comparison principle for $k > 0$ cannot be easily derived due to oscillatory solutions. This is our current project in collaboration with Dr.Changhui Tan (U of S Carolina).

Proof of Theorem

- If $d(0) > \lambda_1(\rho(0) - c_1)$

Proof of Theorem

- If $d(0) > \lambda_1(\rho(0) - c_1)$, we can choose $\eta(0) > \rho(0)$ and $\xi(0) < d(0)$ such that

$$d(0) > \xi(0) \geq \lambda_1(\eta(0) - c_1) > \lambda_1(\rho(0) - c_1).$$

From Proposition for auxiliary system,

$$\xi(t) \geq \lambda_1(\eta(t) - c_1)$$

Proof of Theorem

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From comparison Lemma

$$d(t) > \xi(t) \geq \lambda_1(\eta(t) - c_1) > \lambda_1(\rho(t) - c_1)$$

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Putting together,

$$d(t) > \xi(t) \geq \lambda_1(\eta(t) - c_1) > \lambda_1(\rho(t) - c_1).$$

Euler-Poisson alignment systems

EPA system

Euler Poisson system with alignment induced by kernel ψ:

$$
\begin{aligned}
&\rho_t + (\rho u)_x = 0, \quad x \in \mathbb{T} \\
&u_t + uu_x + \nu u = -k\phi_x + \psi * (\rho u) - u\psi * \rho, \\
&-\phi_{xx} = \rho - c(x).
\end{aligned}
$$

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&-\phi_{xx} = \rho - c(x).
\end{aligned}
$$

Properties of ψ:

- $\psi(x) = \psi(-x),\ x > 0$ (Symmetric),
- $\psi(x+1) = \psi(x)$ (Periodic),
- $|\psi(x) - \psi(y)| \leq K|x - y|$ (Lipschitz continuous).

Tools

- The comparison lemma does not work directly.
- However, a change of variables helps:

$$w = \frac{u_x + \nu + \psi * \rho}{\rho}, \quad s = \frac{1}{\rho}.$$

- The ODE system along characteristic is:

$$\begin{aligned} w' &= k - kcs \\ s' &= w - (\nu + \psi * \rho)s. \end{aligned}$$

- The auxiliary system has two parameters: β and γ:

$$\begin{aligned} p' &= k - k\gamma q \\ q' &= p - \beta q. \end{aligned}$$

Global existence

Theorem 6 (B-Liu '20, Phys D)

If for all $x \in \mathbb{T}$,

$$u_{0x}(x) > \lambda_M \rho_0(x) - \theta_1 - \nu - \psi * \rho_0(x),$$

then there exists a unique global-in-time solution, $\rho, u \in C^1((0,\infty) \times \mathbb{T})$. *Furthermore,*

$$||\rho(t,\cdot)||_\infty \leq ||\rho_0||_\infty e^{\lambda_M t},$$
$$-\lambda_M \leq u_x(t,\cdot) \leq constant(\max u_{0x}, \nu, c_2, \psi_M, \psi_m).$$

$$\psi_m = \min \psi, \quad \psi_M = \max \psi, \quad c_1 = \min c, \quad c_2 = \max c,$$
$$\lambda_M = \frac{(\nu + \psi_M) + \sqrt{(\nu+\psi_M)^2 - 4kc_1}}{2c_1},$$
$$\theta_1 = \frac{-(\nu + \psi_M) + \sqrt{(\nu+\psi_M)^2 - 4kc_1}}{2c_1}.$$

Finite time breakdown

Theorem 7 (B-Liu '20, Phys D)

If $\exists x_0 \in \mathbb{T}$ *such that*

$$u_{0x}(x_0) < \lambda_m \rho_0(x_0) - \theta_2 - \nu - \psi * \rho_0(x_0),$$

then $\exists (t^*, x^*)$ *such that* $\lim_{t \uparrow t^*} u_x(t, x^*) = -\infty$.

$$\lambda_m = \frac{(\nu + \psi_m) + \sqrt{(\nu + \psi_m)^2 - 4kc_2}}{2c_2},$$

$$\theta_2 = \frac{-(\nu + \psi_m) + \sqrt{(\nu + \psi_m)^2 - 4kc_2}}{2c_2}.$$

Thank You!

Questions?