Critical thresholds in spherically symmetric multidimensional Euler-Poisson systems

Manas Bhatnagar
(Joint work with Hailiang Liu (Iowa State University))

ICIAM 2023, Tokyo

Waseda University

August 25, 2023

What's the question?

- Time regularity for convection-dominated problems,

$$\partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla_x)\mathbf{u} = \mathbf{F}, \quad x \in \mathbb{R}^n,$$

to see whether they admit

- Global smooth solution
- Develops shocks/aggregates
- *Critical threshold* phenomena

global regularity $\sim$ a set of initial configurations

shocks/aggregation $\sim$ another set of initial configurations

E. Tadmor, S. Engelberg, H. Liu, J. Carrillo, T. Li, Y. Lee, C. Tan ...
are few of the many researchers interested, working and having significant contributions in this area

Motivation to CTP

Pressureless Euler equation with damping force, $\lambda > 0$,

$$\rho_t + (\rho u)_x = 0, \quad (t, x) \in (0, \infty) \times \mathbb{R},$$
$$u_t + u u_x = -\lambda u,$$

with $u(0, x) = u_0(x)$.

Motivation to CTP

Pressureless Euler equation with damping force, $\lambda > 0$,

$$
\begin{aligned}
&\rho_t + (\rho u)_x = 0, \quad (t,x) \in (0,\infty) \times \mathbb{R}, \\
&u_t + u u_x = -\lambda u,
\end{aligned}
$$

with $u(0,x) = u_0(x)$. Spatial derivative of damped Burgers':

$$(u_x)_t + u(u_x)_x = -u_x^2.$$

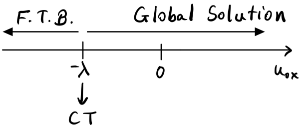
Using method of characteristics,

$$u_x(t, X(t;\alpha)) = \frac{\lambda}{e^{\lambda t}\left((\lambda/u_{0x}(\alpha)) + 1\right) - 1}, \quad \alpha \in \mathbb{R}.$$

Hence, if for some x^*, $u_{0x}(x^*) < -\lambda$, then

$$\lim_{t \to t^*-} u_x = -\infty, \quad t^* = \frac{1}{\lambda} \ln\left(\frac{u_{0x}(x^*)}{\lambda + u_{0x}(x^*)}\right).$$

u_x is bounded for all time if and only if $u_{0x}(x) \geq -\lambda$ for all $x \in \mathbb{R}$.



Criteria of breakdown: $u_x \to -\infty$ (shock formation) along with $\rho \to \infty$ (density concentration) at the same time.

EP system

Multidimensional Euler-Poisson equations,

$$\begin{aligned} &\rho_t + \nabla \cdot (\rho \mathbf{u})_x = 0, \quad x \in \mathbb{R}^N \quad \text{(Mass conservation)} \\ &\mathbf{u}_t + (\mathbf{u} \cdot \nabla)\mathbf{u} = -k\nabla\phi, \quad \text{(Momentum)} \\ &-\Delta\phi = \rho - c(x), \quad \text{(Poisson equation)} \end{aligned}$$

with initial conditions $(\rho_0 \geq 0, u_0) \in H^s(\mathbb{R}^N) \times H^{s+1}(\mathbb{R}^N)$.

- c is the background term which, in practicality, depends on the spatial variable. Need $\int_{\mathbb{T}} \rho_0 - c\, dx = 0$
- k signifies property of underlying forcing: repulsive ($k > 0$) or attractive ($k < 0$).
- Models charge flow in semiconductors, electron flow in ion background ($k > 0$) and gravitation forces in interstellar clouds ($k < 0$)

Spherically symmetric

$$\rho_t + \nabla \cdot (\rho \mathbf{u}) = 0, \quad t > 0, \mathbf{x} \in \mathbb{R}^N,$$
$$\mathbf{u}_t + (\mathbf{u} \cdot \nabla)\mathbf{u} = -k\nabla\phi,$$
$$-\Delta\phi = \rho - c.$$

Spherically symmetric

Symmetry assumption $\rho(t,\mathbf{x}) = \rho(t,r)$, $\mathbf{u}(t,\mathbf{x}) = u(t,r)\frac{\mathbf{x}}{r}$ gives,

$$
\begin{aligned}
&\rho_t + \frac{(\rho u r^{N-1})_r}{r^{N-1}} = 0, \qquad r > 0, \\
&u_t + u u_r = -k\phi_r, \\
&-(r^{N-1}\phi_r)_r = r^{N-1}(\rho - c),
\end{aligned}
$$

with $k > 0$ initial data, $\rho_0(r), u_0(r)$.

Spherically symmetric

Symmetry assumption $\rho(t, \mathbf{x}) = \rho(t, r)$, $\mathbf{u}(t, \mathbf{x}) = u(t, r)\frac{\mathbf{x}}{r}$ gives,

$$
\begin{aligned}
&\rho_t + \frac{(\rho u r^{N-1})_r}{r^{N-1}} = 0, \qquad r > 0,\\
&u_t + u u_r = -k\phi_r,\\
&-(r^{N-1}\phi_r)_r = r^{N-1}(\rho - c),
\end{aligned}
$$

with $k > 0$ initial data, $\rho_0(r), u_0(r)$.

Spherically symmetric

Symmetry assumption $\rho(t,\mathbf{x}) = \rho(t,r)$, $\mathbf{u}(t,\mathbf{x}) = u(t,r)\frac{\mathbf{x}}{r}$ gives,

$$
\begin{aligned}
&\rho_t + \frac{(\rho u r^{N-1})_r}{r^{N-1}} = 0, \qquad r > 0,\\
&u_t + uu_r = -k\phi_r,\\
&-(r^{N-1}\phi_r)_r = r^{N-1}(\rho - c),
\end{aligned}
$$

with $k > 0$ initial data, $\rho_0(r), u_0(r)$.

- $c = 0, u_0 > 0$: Partial results by Engelberg, Liu, Tadmor *(Indiana Univ. J., 2001)*;
  A blow up condition by Yuen *(Non. Anal.: Th., Meth. & Appl., 2011)*, Wei, Tadmor, Bae *(Comm. Math. Sci., 2012)*
- $c = 0$: Sufficient conditions for global existence/breakdown by Tan *(SIMA, 2021)* ;
  $c > 0, d \neq 4$: Rozanova *(Phys. D: Non. Phen., 2023)*, Carrillo, Shu *(ARMA, 2023)*: Subcritical region has zero measure!
- Precise threshold for $c \geq 0$: MB, Liu (2023). More general technique – applicable to both $c = 0$ and $c > 0$

Result for $c > 0$

Theorem (Sharp threshold condition, B-Liu)

If for all $r > 0$, the set of points

$$(r, u_0(r), \phi_{0r}(r), u_{0r}(r), \rho_0(r)) \in \Theta_N,$$

then there is global solution. Moreover, if there exists an $r_c > 0$ such that,

$$(r_c, u_0(r_c), \phi_{0r}(r_c), u_{0r}(r_c), \rho_0(r_c)) \notin \Theta_N,$$

then there is finite-time-breakdown.

$$\Theta_N \subseteq \{(\alpha, x, y, z, \omega) : y/\alpha < c/N, \omega > 0\}.$$

The subcritical region Θ_4

$(\alpha, x, y, z, \omega) \equiv (r, u_0, \phi_{0r}, u_{0r}, \rho_0)$,

$$a := \frac{xz + ky}{\alpha\omega}.$$

A point $(\alpha, x, y, z, \omega) \in \Theta_4$ if and only if the following holds,

- $(\alpha, x, y, z, \omega) \in \{(\alpha, 0, 0, z, \omega) : z^2 < k(2\omega - c)\}$,
- For $a \in \frac{k}{2}\left(\left(\frac{y_m}{\frac{c}{4} - \frac{y}{\alpha}}\right)^{\frac{1}{2}} - 1, \left(\frac{y_M}{\frac{c}{4} - \frac{y}{\alpha}}\right)^{\frac{1}{2}} - 1\right)$,

$$\frac{1}{\max\{\eta_1(0), \eta_2(0)\}} < \omega < \frac{1}{\min\{\eta_1(0), \eta_2(0)\}}, \quad \text{for } x \neq 0,$$
$$z \in \frac{ky}{\alpha a}\left(\frac{d\eta_1}{dt}(0), \frac{d\eta_2}{dt}(0)\right), \quad \text{for } x = 0.$$

Here, $0 < y_m(x, y) < \frac{c}{4} - \frac{y}{\alpha} < y_M(x, y)$,

$\eta_i(t; a, x, y)$ defined through a second order linear IVP.

Preliminaries

- From local existence result: Can extend smooth solutions as long as $||\nabla \mathbf{u}||$ is bounded.
- In the spherically symmetric setup:

$$\nabla\left(u\frac{\mathbf{x}}{r}\right) = \left(u_r - \frac{u}{r}\right)\frac{x \otimes x}{r^2} + \frac{u}{r}\mathbb{I}_N.$$

Preliminaries

- From local existence result: Can extend smooth solutions as long as $||\nabla \mathbf{u}||$ is bounded.
- In the spherically symmetric setup:

$$\nabla \left(u\frac{\mathbf{x}}{r} \right) = \left(u_r - \frac{u}{r} \right) \frac{x \otimes x}{r^2} + \frac{u}{r}\mathbb{I}_N.$$

- Need to bound $u_r, u/r$!

Method of characteristics

$$
\begin{aligned}
&\rho_t + u\rho_x = \dots \\
&(u_r)_t + u\,(u_r)_x = \dots \\
&\left(\frac{u}{r}\right)_t + u\left(\frac{u}{r}\right)_x = \dots \\
&\left(-\frac{\phi_r}{r}\right)_t + u\left(-\frac{\phi_r}{r}\right)_x = \dots .
\end{aligned}
$$

$$
\begin{aligned}
\rho' &= -(N-1)\rho q - p\rho, \\
(u_r)' := p' &= -p^2 - k(N-1)s + k(\rho - c), \\
(u/r)' := q' &= ks - q^2, \\
(-\phi_r/r)' := s' &= -q(c + Ns),
\end{aligned}
$$

with initial data ρ_0, p_0, q_0, s_0.

$q - s$ system

$$(s + c/N) = (s_0 + c/N)exp\left(-N\int_0^t q(\tau)\,d\tau\right)$$

1. Can prove: If $s_0 + c/N > 0$, then q, s periodic

$q - s$ system

$$(s + c/N) = (s_0 + c/N) exp \left(-N \int_0^t q(\tau)\, d\tau \right)$$

1. Can prove: If $s_0 + c/N > 0$, then q, s periodic
2. Can prove: $s_0 + c/N > 0$ (Easy)

$q - s$ system

$$(s + c/N) = (s_0 + c/N) exp \left(-N \int_0^t q(\tau) \, d\tau \right)$$

1. Can prove: If $s_0 + c/N > 0$, then q, s periodic
2. Can prove: $s_0 + c/N > 0$ (Easy)

From Poisson equation:

$$-r^{N-1}\phi_r = \int_0^r \beta^{N-1}(\rho_0(\beta) - c) \, d\beta > -c \int_0^r \beta^{N-1}$$

$$s_0 = \frac{-\phi_{0r}}{r} > -\frac{c}{N}.$$

Trajectory curve

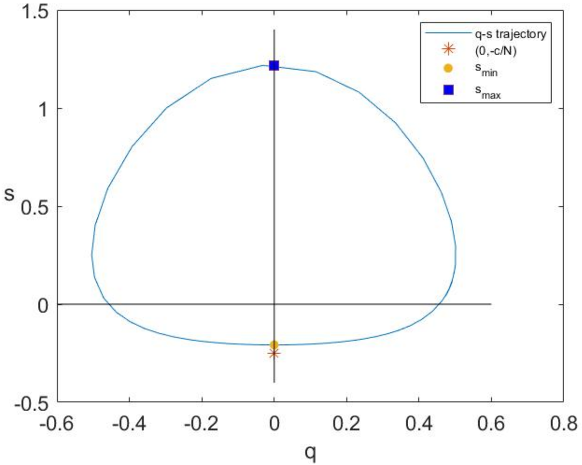


Figure: $N = 4$

Further reduction

$\Gamma = exp(-\int_0^t q)$ (periodic).

$$\left(\frac{1}{\rho}\Gamma^{N-1}\right)' = \frac{p}{\rho}\Gamma^{N-1},$$
$$\left(\frac{p}{\rho}\Gamma^{N-1}\right)' = -k(c+s(N-1))\frac{1}{\rho}\Gamma^{N-1} + k\Gamma^{N-1}.$$

$$\eta' = w,$$
$$w' = -k\eta(c+s(N-1)) + k\Gamma^{N-1}.$$

- p, ρ bounded if and only if $\eta(t) > 0$ for all time.

Final aim: Get a condition on η touching zero.

The quantity A

$$A := qw - k\eta s$$

$$A' + qA = -\left(\frac{k}{N-1}\Gamma^{N-1}\right)',$$

$$A(\Gamma) = \left(A_0 + \frac{k}{N-2}\right)\Gamma - \frac{k}{N-2}\Gamma^{N-1},$$

$$\Gamma(t) = exp\left(-\int_0^t q\right) = \left(\frac{s + c/N}{s_0 + c/N}\right)^{1/N}.$$

The quantity A

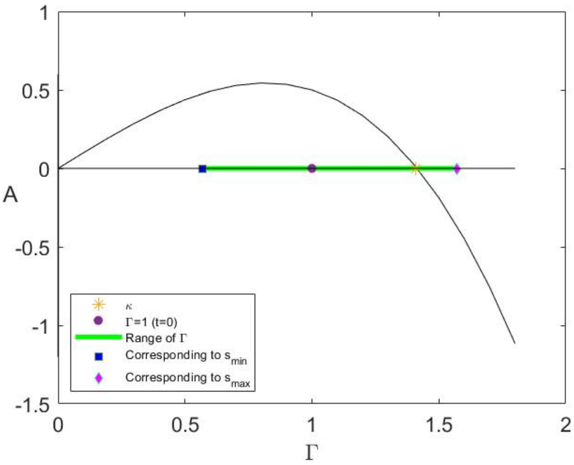
$$A := qw - k\eta s$$

$$A' + qA = -\left(\frac{k}{N-1}\Gamma^{N-1}\right)',$$

$$A(\Gamma) = \left(A_0 + \frac{k}{N-2}\right)\Gamma - \frac{k}{N-2}\Gamma^{N-1},$$

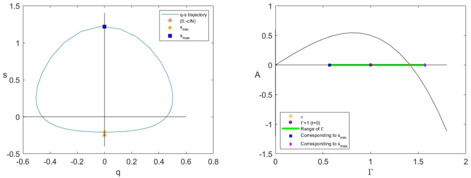
$$\Gamma(t) = exp\left(-\int_0^t q\right) = \left(\frac{s + c/N}{s_0 + c/N}\right)^{1/N}.$$

No explicit expressions of η, w show up in A!!



$$\kappa = \left(1 + \frac{A_0(N-2)}{k}\right)^{1/(N-2)}.$$

Necessary condition for positivity of η



- At time t_* when s (or Γ) is max/min, $q(t_*) = 0$.
- $A(t_*) = qw - k\eta s|_{t=t_*} = -k\eta(t_*)s(t_*)$

Necessary condition: $\kappa \in$ Interior of Range of Γ!

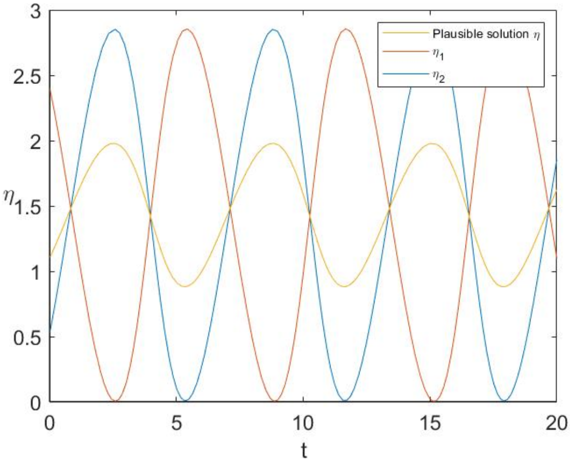
- $t_1, t_2 \in [0, T)$ be the two times such that $A(t_i) = 0$.
- Define η_i as,

$$\eta_i'' + k\eta_i(c + s(N-1)) = k\Gamma^{N-1}, \qquad \eta_i(t_i) = \eta_i'(t_i) = 0.$$

- η_1, η_2 form a cloud around η. (Uniqueness of ODE argument)
- Nonnegativity of η_i's in $[0, T)$.
- η, η_1, η_2 are periodic. (Technical, Floquet's Theorem)
- Consequently, we have positivity of η.

Proposition

η remains in between η_1, η_2 if and only if it is initially so.


Plausible solution η
η_1
η_2
η
t
0
0.5
1
1.5
2
2.5
3
0
5
10
15
20

$c = 0$

- q, s no longer periodic. $(q, s) \to (0, 0)$ and q, s obey (some) convergence rates (Tan, SIMA 21).
- Convergence estimates can be used to arrive at breakdown conditions.
- Can leverage the same techniques as before to construct bounds on η.
- ODE comparison principle can be used to obtain global existence for different cases.

$c = 0$

- For $u_0 > 0$: (Tedious) calculations result in closed expression of the threshold condition in terms of initial data ρ_0, p_0, q_0, s_0,

$$\rho_0 < F_N(\rho_0, p_0, q_0, s_0),$$

  for a super-complicated F_N.
- Can recover the simpler condition for $N = 4$ as obtained by Liu, Engelberg, Tadmor (2001).

$c = 0$

- For $u_0 > 0$: (Tedious) calculations result in closed expression of the threshold condition in terms of initial data ρ_0, p_0, q_0, s_0,

$$\rho_0 < F_N(\rho_0, p_0, q_0, s_0),$$

  for a super-complicated F_N.
- Can recover the simpler condition for $N = 4$ as obtained by Liu, Engelberg, Tadmor (2001).

A cute sufficient condition for breakdown for $c = 0$:

Theorem (Sufficient condition for breakdown, B-Liu)

Suppose $c = 0$ and $N \geq 3$. If there exists an $r_c > 0$ such that,

$$A_0(r_c) \leq -\frac{k}{N-2},$$

then there is finite-time-breakdown.

Threshold for $c = 0$

$(\alpha, x, y, z, \omega) \equiv (r, u_0, \phi_{0r}, u_{0r}, \rho_0)$,
A point $(\alpha, x, y, z, \omega) \in \Theta_N$ if and only if the following holds,

- For $a \in \left(-\frac{k}{N-2}, 0\right)$,

$$z > -\frac{ky}{x} + \frac{\alpha a}{x\eta_1(0)}, \quad \text{for } x \neq 0, \qquad z > \frac{ky}{\alpha a}\frac{d\eta_1}{dt}(0), \quad \text{for } x = 0.$$

- For $a = 0$,

$$z = -\frac{ky}{x},\ \omega > \frac{1}{\eta_2(0)}, \quad \text{for } x < 0, \qquad z = -\frac{ky}{x},\ \omega > 0, \quad \text{for } x > 0.$$

- For $a \in \left(0, \frac{k}{N-2}\left(\left(\frac{-\alpha y_N}{y}\right)^{1-\frac{2}{N}} - 1\right)\right)$,

$$-\frac{ky}{x} + \frac{\alpha a}{x\eta_1(0)} < z < -\frac{ky}{x} + \frac{\alpha a}{x\eta_2(0)}, \quad \text{for } x < 0, \qquad \omega > 0, \quad \text{for } x > 0.$$

- For $a \in \left[\frac{k}{N-2}\left(\left(\frac{-\alpha y_N}{y}\right)^{1-\frac{2}{N}} - 1\right), \infty\right)$,

$$\omega > 0, \quad \text{for } x > 0.$$

- $0 < -\frac{y}{\alpha} < y_N = y_N\left(\frac{u_0(\alpha)}{\alpha}, \frac{\phi_{0\alpha}(\alpha)}{\alpha}\right) = \left(\frac{(N-2)}{2k}\right)^{\frac{N}{N-2}} \left(R_N\left(\frac{u_0(\alpha)}{\alpha}, \frac{-\phi_{0\alpha}(\alpha)}{\alpha}\right)\right)^{\frac{N}{N-2}}.$
- The explicit expression for $R_N(\cdot,\cdot)$ can be computed.
- The functions

$$\eta_i = \eta_i\left(t; \frac{u_0(\alpha)}{\alpha}, \frac{\phi_{0r}(\alpha)}{\alpha}, A_0(\alpha)\right), \quad i = 1, 2,$$

  are defined via linear IVPs, similar to the $c > 0$ case.

Future Work

- Apply our techniques to the Euler-Poisson-alignment system.
- Apply our techniques to Euler-Poisson system with swirl.

References

- A complete characterization of sharp thresholds to spherically symmetric multidimensional pressureless Euler-Poisson systems,
  MB, Hailiang Liu
  https://arxiv.org/abs/2302.04428

Registration and travel support for this presentation was provided by National Science Foundation Grant DMS-2233032 and the Society for Industrial and Applied Mathematics.

THANK YOU