Critical Thresholds in 1D Damped Euler-Poisson Systems

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Abstract

We use a transformation of variables to come up with the precise threshold curves for pressureless Euler-Poisson system of equations in 1D for all three cases of damping: overdamped, underdamped and borderline damped. We represent the sharp thresholds as the boundary between two invariant sets in a plane through a uniquely defined curve. Subsequently, we apply these general results to recover thresholds in a non-local system [1] relevant for modeling collective behavior in mathematical biology.

Introduction

We are concerned with the critical threshold phenomena in the following Euler-Poisson system,

$$
\begin{aligned}
&\rho_t + (\rho u)_x = 0,\\
&u_t + uu_x = -k\phi_x - \nu u,\\
&-\phi_{xx} = \rho - c,
\end{aligned}
\tag{1}
$$

on $\mathbb{R} \times (0, \infty)$ subject to C^1 initial conditions. Here $c > 0$ is the constant background, $\nu > 0$ is the damping coefficient, and k is the force direction coefficient. $k > 0$ signifies repulsive forces and vice-versa. Finite time breakdown vs global regularity of the solution is an important aspect of such systems. This question has been addressed in a series of works.

Key Idea

Along the particle path,
$\Gamma = \{(x,t) | \, x'(t) = u(x(t), t), x(0) = \alpha \in \mathbb{R}\}$, the dynamics of ρ and $d := u_x$ are governed by,

$$
\begin{aligned}
&\rho' + \rho d = 0,\\
&d' + d^2 + \nu d = k(\rho - c).
\end{aligned}
$$

We introduce a change of variables,

$$
r = -\frac{d}{\rho}, \quad s = \frac{1}{\rho}.
$$

Consequently, we obtain the following linear system,

$$
\begin{aligned}
&r' = -\nu r - k(1 - cs),\\
&s' = -r,
\end{aligned}
\tag{2}
$$

with appropriate initial conditions. We employed the method of characteristics to convert the PDE system (1) to ODE system (2). On observation, $s(t) > 0 \; \forall t > 0 \iff$ global solution. This linear ODE system can be uniquely solved to obtain the critical threshold. We further characterize the threshold curve using a uniquely defined curve.

Results

Theorem

For the given 1D Euler Poisson system (1), $\exists (x^*, t^*)$ for which $\lim_{t \to t^*-} u_x(x^*, t) = -\infty$ iff $\exists x \in \mathbb{R}$ such that

1. *($\nu > 2\sqrt{kc}$) Strong damping*

$$
\max\{u_0'(x), \; u_0'(x) + \lambda_2(c - \rho_0(x))\} < 0, \; \textit{and}
$$
$$
\left| \frac{c\lambda_2 u_0'(x) + k(c - \rho_0(x))}{k\rho_0(x)} \right|^{\lambda_1} \leqslant \left| \frac{c\lambda_1 u_0'(x) + k(c - \rho_0(x))}{k\rho_0(x)} \right|^{\lambda_2},
$$

where $\lambda_1 = \frac{\nu - \sqrt{\nu^2 - 4kc}}{2c}$ *and* $\lambda_2 = \frac{\nu + \sqrt{\nu^2 - 4kc}}{2c}$.

2. *($\nu = 2\sqrt{kc}$) Borderline damping*

$$
\max\{u_0'(x), u_0'(x) + \frac{\nu}{2c}(c - \rho_0(x)\} < 0, \; \textit{and}
$$
$$
\ln\left(-\frac{2cu_0'(x) + \nu(c - \rho_0(x))}{\nu\rho_0(x)} \right) \geqslant \frac{2cu_0'(x)}{2cu_0'(x) + \nu(c - \rho_0(x))}.
$$

3. *($\nu < 2\sqrt{kc}$) Weak damping*

$$
\left(u_0'(x) + \frac{\nu(c - \rho_0(x))}{2c} \right)^2 \geqslant \mu^2 \left[\frac{\rho_0^2(x)}{c^2} \left(\frac{kce^{\nu t^*}}{\mu^2} - 1 \right) + \frac{(2\rho_0(x) - c)}{c} \right],
$$

where $\mu = \sqrt{kc - \nu^2/4}$ *and*

$$
t^* = \frac{1}{\mu} \left[\beta + \tan^{-1} \left(\frac{2\mu u_0'(x)}{\nu u_0'(x) + 2k(c - \rho_0(x))} \right) \right],
$$
$$
\beta = \begin{cases} 0 & \nu u_0'(x) < \min\{0, 2k(\rho_0(x) - c)\},\\ \pi & 2k(\rho_0(x) - c) < \nu u_0'(x),\\ 2\pi & 0 < \nu u_0'(x) < 2k(\rho_0(x) - c). \end{cases}
$$

Theorem

For the Euler-Poisson system (1), there exists a unique solution $\rho, u \in C^1(\mathbb{R} \times (0, \infty))$ iff $\forall x \in \mathbb{R}$,

1. ***($\nu > 2\sqrt{kc}$) Strong damping:*** $(\rho_0(x), u_0'(x)) \in \{(\rho, d) : d > -\rho Q_a(1/\rho), \; \rho > 0\}$,
where $Q_a : [0, \infty) \longrightarrow [0, \infty)$ is a continuous function satisfying

$$
\frac{dQ_a}{ds} = \nu + \frac{k}{Q_a}(1 - cs), \; Q_a(0) = 0.
$$

2. ***($\nu = 2\sqrt{kc}$) Borderline damping:*** $(\rho_0(x), u_0'(x)) \in \{(\rho, d) : d > -\rho Q_b(1/\rho), \; \rho > 0\}$,
where $Q_b : [0, \infty) \longrightarrow [0, \infty)$ is a continuous function satisfying

$$
\frac{dQ_b}{ds} = 2\sqrt{kc} + \frac{k}{Q_b}(1 - cs), \; Q_b(0) = 0.
$$

3. ***($\nu < 2\sqrt{kc}$) Weak damping:***

$$
(\rho_0(x), u_0'(x)) \in \{(\rho, d) : -\rho Q_1(1/\rho) < d < -\rho Q_2(s^* - 1/\rho), \; \rho \in (1/s^*, \infty)\},
$$

where $s^* > 0$ is uniquely determined and $Q_1 : [0, s^*] \longrightarrow \mathbb{R}^+ \cup \{0\}$ and $Q_2 : [0, s^*] \longrightarrow \mathbb{R}^- \cup \{0\}$ are continuous functions satisfying

$$
\frac{dQ_1}{ds} = \nu + \frac{k}{Q_1}(1 - cs), \; Q_1(0) = 0,
$$
$$
\frac{dQ_2}{ds} = -\nu + \frac{k}{Q_2}(c(s^* - s) - 1), \quad Q_2(0) = 0.
$$

Application to an aggregation system [1]

Instead of electric force governed by the Poisson equation, here we have non-local interactions between particles. The system then reads,

$$
\begin{aligned}
&\rho_t + (\rho u)_x = 0, \quad (x,t) \in \mathbb{R} \times (0, \infty),\\
&u_t + uu_x = -\partial W \star \rho - u,
\end{aligned}
\tag{3}
$$

where

$$
W(x) = -|x| + \frac{|x|^2}{2},
$$

subject to C^1 initial conditions with $\int_{\mathbb{R}} \rho_0(x) \, dx = M_0$.

Theorem

For the 1D pressureless damped Euler system of equations (3), there exists a unique global solution $\rho, u \in C^1(\mathbb{R} \times (0, \infty))$ iff $\forall x \in \mathbb{R}$,

1. *(Subcritical mass $M_0 < 1/4$)*

$$
(\rho_0(x), u_0'(x)) \in \{(\rho, d) : d > -\rho R_a(1/\rho), \; \rho > 0\},
$$

where $R_a : [0, \infty) \longrightarrow [0, \infty)$ is a continuous function satisfying

$$
\frac{dR_a}{ds} = 1 + \frac{1}{R_a}(2 - M_0 s), \; R_a(0) = 0.
$$

2. *(Critical mass $M_0 = 1/4$)*

$$
(\rho_0(x), u_0'(x)) \in \{(\rho, d) : d > -\rho R_b(1/\rho), \; \rho > 0\},
$$

where $R_b : [0, \infty) \longrightarrow [0, \infty)$ is a continuous function satisfying

$$
\frac{dR_b}{ds} = 1 + \frac{1}{R_b}(2 - s/4), \; R_b(0) = 0.
$$

3. *(Supercritical mass $M_0 > 1/4$)*

$$
(\rho_0(x), u_0'(x)) \in \{(\rho, d) : -\rho R_1(1/\rho) < d < -\rho R_2(\gamma - 1/\rho), \; \rho \in (1/\gamma, \infty)\},
$$

where

$$
\gamma = \frac{2}{M_0}\left(1 + e^{\frac{\pi}{\sqrt{4M_0 - 1}}}\right),
$$

and $R_1 : [0, \gamma] \longrightarrow \mathbb{R}^+ \cup \{0\}$ is a continuous function satisfying

$$
\frac{dR_1}{ds} = 1 + \frac{1}{R_1}(2 - M_0 s), \; R_1(0) = 0,
$$

and $R_2 : [0, \gamma] \longrightarrow \mathbb{R}^- \cup \{0\}$ is another continuous function satisfying

$$
\frac{dR_2}{ds} = -1 + \frac{1}{R_2}(M_0(\gamma - s) - 2), \quad R_2(0) = 0.
$$

References

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