A Convergent Front Tracking Scheme, with an Application to Gas Dynamics

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NCTS International Conference on Shock Wave Theory in General Relativity and Fluid Dynamics

Apr 9, 2026

Hyperbolic conservation laws

In one spatial dimension,

$$q(U)_t + f(U)_x = 0, \quad \mathbb{R} \times (0, \infty),$$

where

- $\mathcal{U}$ is the state space of unknown vector,
- $q, f : \mathcal{U} \to \mathbb{R}^N$ are C^2 functions with Dq invertible,
- the eigenvalue problem $Df = \lambda Dq$ has real eigenvalues,
- strictly hyperbolic if $\lambda_1(U) < \lambda_2(U) < \ldots < \lambda_N(U)$.

N characteristic fields: i-th field is either genuinely nonlinear $D\lambda_i \cdot r_i > 0$ or linearly degenerate $D\lambda_i \cdot r_i \equiv 0$.

Weak/Weak* solution

$U \in L^\infty_{loc}$ is a weak solution on $[0, T)$ if

$$\int_0^T \int_{\mathbb{R}} q(U) \cdot \psi_t + f(U)\psi_x \, dx \, dt + \int_{\mathbb{R}} U(x,0)\, \psi(x,0)\, dx = 0,$$

for all $\psi \in C_c^\infty(\mathbb{R} \times [0, T))$.

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for all $\psi \in C_c^\infty(\mathbb{R} \times [0, T))$.
Miroshnikov-Young '17: Equivalent definition by treating the solution as an element of the dual of the test functions:

$$q(U(\cdot,t)) - q(U_0) + * \int_0^t \mathbb{D}f(U(\cdot,\tau))\, d\tau = 0,$$

in X^*, where $* \int$ is the Gelfand Integral,

$$\left\langle * \int_E g(\cdot,y)\, dy, \psi(\cdot) \right\rangle = \int_E \langle g(\cdot,y), \psi(\cdot)\rangle \, dy.$$

Equivalently: $q(U)' + \mathbb{D}f(U) = 0$, a.e (0,T) , with $U(\cdot,0) = U_0$.

Weak* solution

Theorem (Miroshnikov-Young '17)

Under mild growth assumptions, weak and weak solutions are equivalent.*

- In 1D, it is widely accepted that the correct space to look for solutions is the space of functions of bounded variation,

$$Var(g) := \sup \sum_i |g(x_i) - g(x_{i-1})| < \infty.$$

Theorem (J. Glimm '65)

If $||U_0||_\infty + Var(U_0) \ll 1$ *then there is an 'entropic' weak solution with*

$$||U||_\infty \leq C||U_0||_\infty, \qquad Var(U(\cdot, t)) \leq CVar(U_0).$$

- Hence, we consider $X = C(K), X^* = \mathcal{M}(K)$.

Subsequent results

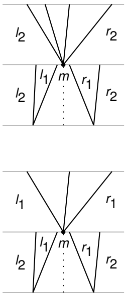
- Glimm, Lax '70: For $N = 2$, along with a geometrical assumption, $||U_0||_\infty \ll 1$ implies existence of 'entropy' solution. The geometry assumption was removed by Bianchini, Colombo, Monti '09.
- Young '93:Sup-norm stability for general N.
- Bressan, Goatin '99: Uniqueness under 'Tame Oscillation Condition'.
- Chen, Krupa, Vasseur '22: L^2 stability and uniqueness.
- DiPerna '76; Risebro, Tveito '92; Bressan '92; Baiti, Jenssen '98 Solution as limit of **Front Tracking Approximations**.

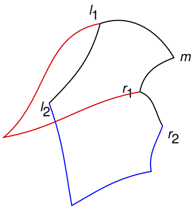
All of the above are in the realm of weak waves $||U||_\infty \ll 1$.

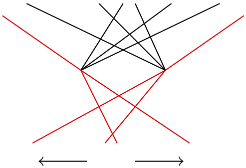
Weak shocks to strong shocks

Key idea for existence: $\Delta(V + P) \leq 0$ on a discrete scheme.

- Interdependency of the scheme and functional
- Occurence of vacuum in the discrete scheme



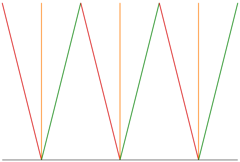




Credits: Alberto Bressan

Idea of FT

- For initial data U_0, let $\widehat{U_0}$ be its piecewise constant approximation in L^1_{loc} (equivalently M_{loc}).
- At each discontinuity, solve an RP and evolve.
- When two fronts (discontinuities) meet, solve an RP.



Initialization

- Hence, at each time the function $\widehat{U}_\epsilon(\cdot, t)$ is piecewise constant.
- In particular, $\widehat{U}_\epsilon \in W^{1,\infty}([0,T); BV, \mathcal{M})$.

Residual

For any function $g \in W^{1,\infty}_{*}(0, T; BV, \mathcal{M})$, define

$$\mathcal{R}(g)(t) = q(g(\cdot, t))' + \mathbb{D}f(g(\cdot, t)),$$

which is time-dependent measure.

Residual

For any function $g \in W_*^{1,\infty}(0, T; BV, \mathcal{M})$, define

$$\mathcal{R}(g)(t) = q(g(\cdot, t))' + \mathbb{D}f(g(\cdot, t)),$$

which is time-dependent measure. Say we have a family of approximate solutions $\widehat{U}_\epsilon$ constructed through mFT, with the following

1. $\widehat{U}_\epsilon$ are (for any fixed time) piecewise constant functions,
$$\widehat{U}_\epsilon = U_0 + \sum_k [U_\epsilon]_k H(x - x_k - s_k(t - t_k)).$$
2. $$Var(\widehat{U}_\epsilon(\cdot, t)) \leq K.$$
3. $$\lim_{\epsilon \to 0^+} ||\mathcal{R}(\widehat{U}_\epsilon)||_{L^\infty(0,T;M)} = 0.$$

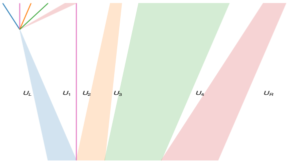
(2) implies uniform boundedness of $\widehat{U}_\epsilon, \mathbb{D}\widehat{U}_\epsilon$ in $L^\infty(0, T; M) = L^1(0, T; C)^*$, which by Banach-Alaoglu gives a limit.
(3) ensures limit is a solution.
We will see that for mFT solutions (2) $\Longrightarrow$ (3).

Our philosophy for mFT

- For all nonlinear waves, adjacent states defining **all waves** must be treated **exactly**;
- To correctly resolve interactions of strong waves, **compressions** must be treated **explicitly**;
- For finite times, the **number of waves** in the approximation must remain **finite**;
- For general systems, the scheme should be as **flexible** as possible.

Generalized RP (gRP)

- Given left and right states U_l, U_r along with N nonnegative widths $w_1, \ldots, w_N$ with $w_i = 0$ for every linearly degenerate family i.
- The problem is solved by piecing together, in increasing order of wavespeed, **either** shocks/contacts in case $w_k = 0$, **or** centered simple waves of initial width $w_k > 0$.



- gRS is an **exact** solution up to the first time of interaction.

Solvability of gRS

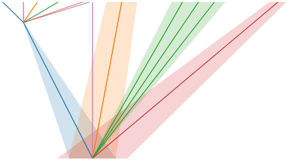
- **Assumption**: gRS is globally uniquely solvable in $\mathcal{U}$. That is, given states U_L, $U_R \in \mathcal{U}$ and initial widths w_i, the equation $u_N = u_R$ has a unique solution $\{\zeta_k\}$, in which each U_i is connected to U_{i+1} as mentioned.
- **Extended Lax condition**: Given any $U_m \in \mathcal{U}$,

$$\sigma_i(U_l, U_m) < \sigma_{i'}(U_m, U_r)$$

for all $i < i'$, U_l lying on i-th (inverse) shock curve through U_m and U_r on the i'-th shock curve through U_m.
- EL condition is global and maintains the order of the waves.
- EL$+$Geometric assumption on shock/simple wave curves implies existence and uniqueness.

Discretized gRS (dgRS)

- Given left and right states U_l, U_r along with N nonnegative widths $w_1, \ldots, w_N$ with $w_i = 0$ for every linearly degenerate family i.
- In dgRS, all waves emanate from a single point and the widths are virtual, that is, information that is carried with the wave and not explicitly visible in the scheme.



The residual of the dgRS approximation is:

$$\mathcal{R}(U_d) = \sum_k \left([f]_k - s_k[q]_k\right)\delta_{s_k t}.$$

Some key elements of mFT

1. Each wave γ_k carries additional data with itself: left/right states, its speed, width at the time of generation, time of generation, center of (simple) wave, time of interaction.
2. Owing to our philosophy of keeping states exact, we can modify speeds and widths only.
3. 'All' waves are given least square wavespeeds:

$$s_k := \text{argmin}|[f]_k - s[q]_k|^2 = \frac{[f]_k \cdot [q]_k}{[q]_k \cdot [q]_k}$$

4. If wave is a shock, then $s_k = \sigma$, which is the R-H speed.

Lemma

If $U_r = W(U_l, \zeta)$, then

$$R_* := \mathcal{R}(([q]_k + [f]_k)H) = \left(I - \frac{[q][q]^T}{[q]^T[q]} \right) [f].$$

Moreover, if $\zeta \ll 1$, then $R_* = O(1)|\zeta_k|^3$.

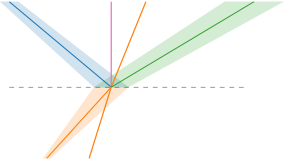
Convergence

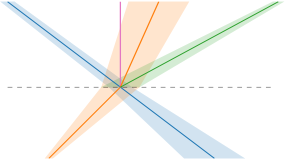
Although the scheme will be modified that would change the following convergence argument, the basic idea is:

$$
\begin{aligned}
||\mathcal{R}(\widehat{U}_\epsilon)||_{\mathcal{M}} &= \left|\left|\mathcal{R}\left(U_0 + \sum_k [U_\epsilon]_k H(x - x_k - s_k(t - t_k))\right)\right|\right|_{\mathcal{M}} \\
&= \left|\left|\sum_k ([f]_k - s_k[q]_k)\delta_k\right|\right| \\
&= \sum_k |[f]_k - s_k[q]_k| \\
&= \sum_{\text{Simple waves}} |[f]_k - s_k[q]_k| \\
&\leq O(1) \sum_{\text{Simple waves}} |\zeta_k|^3 \\
&\leq \epsilon^{2p_s} Var(\widehat{U}_\epsilon(\cdot, t)) \leq K\epsilon^{2p_s}.
\end{aligned}
$$

Of course, we assumed the uniform BV bounds.

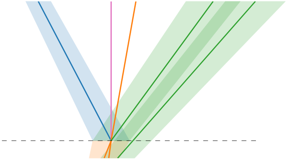
Interactions: Each interaction is solved as a dgRP.
Waves of same family: One of them must be a shock. In this case, the outgoing wave of this family is given zero width and all others are given the width of the other wave.
Crossing waves: Each of them retain their widths, and the other waves are given the width which is max of the two.

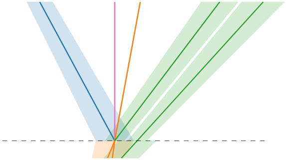




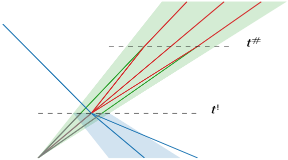
Accelerated collapse of compressions: Compressions collapse to emit new waves. Extremely weak compressions add weaker waves. Therefore, in a dgRS if a very weak compression is emitted, we set its width to zero and resolve the dgRP. Similarly, we perform this collapse for a strong compression.

Avoiding overlaps: Immediately after each interaction (dgRS), we check for any adjacent simple wave overlaps and remove them. We enforce this condition inductively: first upon initialization, and then by restricting virtual wave widths as necessary when resolving interactions.



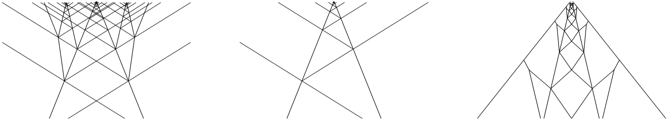


Multirarefactions: Mechanism to split rarefactions which have some finite width at the time of generation



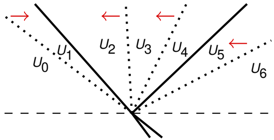
Composite Waves

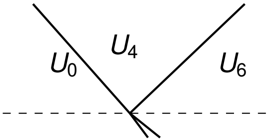
- A serious issue with FT schemes is an accumulation of interaction times.



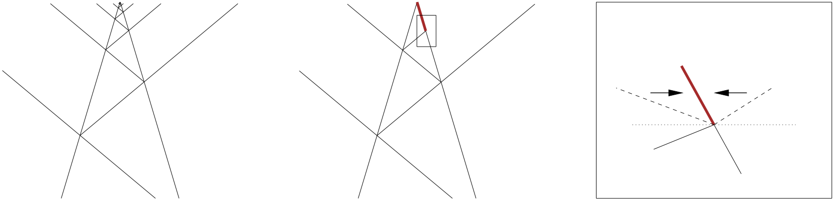
$N = 4$, a nonphysical system with 3 families (Jenssen '00) and a nonphysical system

- Recall our philosophy of keeping states exact, so we do not use approximate Riemann Solver.
- Solve dgRS exactly and identify **carrier waves** (relatively strong) and **passengers** (weaker)
- Use the virtual width of the base wave as that of the composite.
- In this way, the smaller components 'piggyback' onto the carrier.





- Any wave stronger than ϵ^{p_p} cannot be a passenger.
- If all generated waves are weaker, then the strongest of the all is the base. All others are passengers.
- We call this a **lonely weak wave**.
- If a dgRS generates an lww, then that is the only wave generated at that interaction.
- Consequently, we do not allow any crossing of waves.
- Multirarefactions can never be passengers.
- The speed of the carrier is the speed of the wave.



Initialization in coherence with mFT

1. Explicitly note finite discontinuities in the data by setting

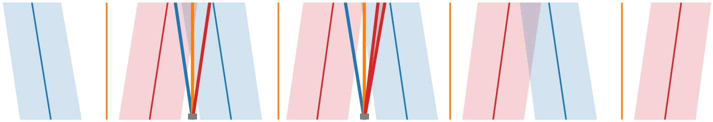
$$X_d := \{x \ : \ |u^0(x+) - u^0(x-)| \geq \epsilon^{2f}\},$$

2. Between any consecutive points $\{x_\flat, x_\sharp\}$ of X_d, there are finitely many points,

$$x_\flat =: x_0 < x_1 < \cdots < x_{m-1} < x_m := x_\sharp,$$

3. $$\sum_{x_\sharp, x_\flat} \left| \|u^0\|_{BV(x_\flat+, x_\sharp-)} - \sum_{j=1}^{m} |u^0(x_j) - u^0(x_{j-1})| \right| < \epsilon,$$

4. gRS is solved between x_k, x_{k+1} with user-defined (positive) widths, hence no shocks. Then each simple wave is replaced by a single jump.
5. RP solved at elements of X_d.



α-ray condition

- Fix a point (x_*, t_*), where t_* is in the closure of the times up to where the scheme can be continued.
- Consider the trace of the approximation on the ray with some speed α and ending at (x_*, t_*), that is

$$V_\alpha(t) := \widehat{U}^\epsilon\big(x_* + \alpha\,(t - t_*), t\big), \qquad t \in [0, t_*),$$

- Since mFT approximation is a sum of Heavisides,

$$V_\alpha(t) := \sum_j [v]_j\, H(t - t_j), \quad 0 \le t < t_*, \text{where}$$

$$[v]_j := U^\epsilon\big(x_* + \alpha\,(t_j - t_*), t_j +\big) - U^\epsilon\big(x_* + \alpha\,(t_j - t_*), t_j -\big).$$

- There could be infinitely many jumps in V_α.

Condition: The following limit exists, uniformly for α in compact intervals:

$$V_\alpha := \lim_{t \to t_*^-} V_\alpha(t)$$

Continuation of scheme

Theorem (B-Young, 2025)

For fixed $\epsilon > 0$, suppose that the mFT approximation $\widehat{U}^{\epsilon}$ satisfies the α-ray condition for all (x, t), with $t \leq t_{\infty}$. Then there are no accumulating trajectories and the scheme can be continued for all times, that is $t_{\infty} = \infty$.

Continuation of scheme

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Corollary

If the total variation of the mFT approximation blows up in finite time,

$$\left\| U^\epsilon(\cdot, t) \right\|_{BV} \to \infty \text{ as } t \to t_\infty,$$

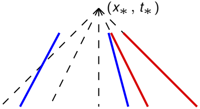
then the α-ray condition fails at some point (x_∞, t_∞).

Idea of proof

- The proof goes by contradiction. Let (x_*, t_*) be the accumulation point.
- By the α-ray condition, for all $k, k' \geq K$, and $\alpha \in [-c_m, c_m]$,

$$\left|\widehat{U}^{\epsilon}\left(x_\infty + \alpha\left(t_k - t_\infty\right), t_k\right) - \widehat{U}^{\epsilon}\left(x_\infty + \alpha\left(t_k - t_\infty\right), t_{k'}\right)\right| < \frac{1}{2}\,\epsilon^{p_p},$$

where ϵ^{p_p} is the maximum strength of any passenger.



Blue is lww, red is not.

Convergence

Elements to look out for:

- Multirarefactions
- Composite waves

For composite waves: heavy passengers and light passengers,

$$|\zeta_j| \leq Q(\epsilon)\,|\zeta_{b_j}|, \text{ or}$$

$$Q(\epsilon)\,|\zeta_{b_j}| < |\zeta_j| \leq |\zeta_{b_j}|,$$

Quantification of ‘amount’ of heavy passengers:

$$W_H(\Gamma,\epsilon) := \sum_{j\in B_H} \sqrt{\sum_{b_l=j} \zeta_l^2},$$

Lemma

For fixed t, the residual of the wave sequence $\Gamma = \Gamma^t$ satisfies the estimate

$$\left\|\mathcal{R}(\Gamma)\right\|_M \leq K_l\,Var(\Gamma^t)\left(\epsilon^2 + Q(\epsilon)\right) + K_{hp}\,W_H(\Gamma,\epsilon),$$

where $K_\square$ are constants depending only on the system via a compact region of state space $\mathcal{U}$.

Idea of proof

1. Composite waves with heavy and light passengers treated separately.
2. Light passengers can be controlled by the strength of the base. For γ_j with $j \in B_L$,

$$
\begin{aligned}
\mathcal{R}(\gamma_j) &= \mathcal{R}(\gamma_b) + \mathcal{R}\left(\sum_{j:b_j=b} \gamma_j\right) \\
&\lesssim |\zeta_b|^3 + \sum |\zeta_j| \\
&\lesssim |\zeta_b|^3 + (N-1)Q(\epsilon)|\zeta_b| \\
&\lesssim |\zeta_b|\left(\epsilon^2 + Q\right)
\end{aligned}
$$

3. Assuming uniform BV bounds and a consequent application of Banach-Alaoglu gives us an appropriate limit...

Final result for general systems

Theorem (B,Young '25)

Suppose the mFT scheme is defined for $t \le T$ with the uniform variation bound

$$\mathcal{V}(\Gamma^t) < V, \quad \textit{for } 0 \le t \le T.$$

Suppose also that $Q(\epsilon)$ can be chosen so that

$$Q(\epsilon) = o(1),$$
$$W_H(\Gamma, \epsilon) = o(1),$$
$$\epsilon \to 0^+,$$

uniformly for $t \in [0, T]$. Then some subsequence of the corresponding approximations $\widehat{U}^\epsilon$ converge as $\epsilon \to 0^+$, and the limit is an entropic weak solution.*

3×3 Gas Dynamics

- v, u, p, s are the specific volume, velocity, pressure and specific entropy,

$$(-v(p,s))_t + u_x = 0, \qquad u_t + p_x = 0, \qquad \left(\tfrac{1}{2}\,u^2 + h - pv\right)_t + (u\,p)_x = 0,$$

where $h = h(p,s)$ is the enthalphy satisying, $v = \frac{\partial h}{\partial p}, \theta = \frac{\partial h}{\partial s}$.

$$U := [p\ u\ s]^T.$$

$$Dq(U) = \begin{bmatrix} -\frac{\partial v}{\partial p} & 0 & -\frac{\partial v}{\partial s} \\ 0 & 1 & 0 \\ -p\,\frac{\partial v}{\partial p} & u & \theta - p\,\frac{\partial v}{\partial s} \end{bmatrix}, \qquad Df(U) = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ u & p & 0 \end{bmatrix}.$$

Forward and backward wave curves

$$u_r - u_l = \pm \int_{p_l}^{p_r} \frac{1}{c(w,s)}\,dw, \qquad s_r = s_l,$$

and middle family (entropy jumps): $u_r = u_l, p_r = p_l$.

$$u_r - u_l = \frac{1}{\sigma}(p_r - p_l), \qquad \sigma = \pm\sqrt{\frac{p_r - p_l}{v_l - v_r}},$$

Lemma

The gRP for the Euler equations is globally uniquely solvable, although possibly with a vacuum.

- Global solvability and uniqueness of RP is widely known.
- Given $U_l =: U_0, U_4 := U_r$, we need to find U_1, U_2, U_3 that satisfy

$$U_1 - U_0 = (p_0 - p_1)\,\overline{r}_-,\ U_2 - U_1 = (s_2 - s_1)\,\overline{r}_0,\ U_4 - U_3 = (p_3 - p_2)\,\overline{r}_+,$$

$$\overline{r}_- := \begin{pmatrix} -1 \\ 1/\{c\} \\ [s]/[p] \end{pmatrix}, \qquad \overline{r}_0 := \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \qquad \overline{r}_+ := \begin{pmatrix} 1 \\ 1/\{c\} \\ [s]/[p] \end{pmatrix},$$

$$\frac{1}{\{c\}} := \left\langle \frac{1}{c} \right\rangle \text{ or } \{c\} := |\sigma|, \qquad [s] = 0 \text{ for simple}$$

Equivalence of Material and Spatial frames

In spatial frame,

$$\rho_t + (\rho\, u)_y = 0,$$
$$(\rho\, u)_t + \left(\rho\, u^2 + p\right)_y = 0,$$
$$\left(\tfrac{1}{2}\,\rho\, u^2 + \rho\, e\right)_t + \left(\tfrac{1}{2}\,\rho\, u^3 + \rho\, u\, e + u\, p\right)_y = 0.$$

$\rho v = 1, e = h - pv$. Smooth solutions are related by the transformation

$$x := \int^y \rho(w,t)\, dw, \text{ so that } \frac{\partial x}{\partial y} = \rho(y,t).$$

Wagner '87: Equivalence of weak solutions

Lemma

The mFT schemes defined using either the Lagrangian or Eulerian frames are equivalent. That is, the bi-Lipschitz maps from one to the other frame respect both the mFT scheme and the limits thereof.

Few more results

- To obtain the previous Lemma, we derive a relation between the wave/shock speeds:

$$\lambda_i^E(U) = u + v\,\lambda_i^L(U), \qquad \sigma_i^E(U) = \overline{u} + \overline{v}\,\sigma_i^L(U),$$

Preconditioner: Each γ_j tags along a matrix A_j so that

$$\mathcal{R}^P(\gamma_j) = A_j \mathcal{R}(\gamma_j).$$

Preconditioned wave speed:

$$s_j^P := \operatorname{argmin} |A_j([f]_j - s[q]_j)|^2.$$

Set $\beta = \left\langle \frac{1}{c} \right\rangle^2 \Big/ \left\langle \frac{1}{c^2} \right\rangle$ and choose,

$$A := \begin{pmatrix} \left\langle \frac{1}{c} \right\rangle \Big/ \left\langle \frac{1}{c^2} \right\rangle & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -\overline{p} & -\overline{u} & 1 \end{pmatrix}$$

Lemma

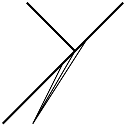
The (preconditioned) least squared wavespeed s^P and the ratio β for a simple wave satisfy

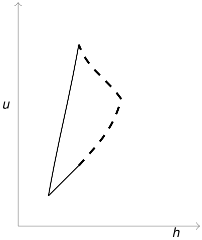
$$s^P = \pm \frac{\beta + 1}{2\langle 1/c \rangle} + O(|[p]|^2), \text{ and } \beta = 1 + O(|[p]|^2),$$

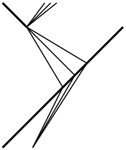
respectively, and the corresponding residual satisfies

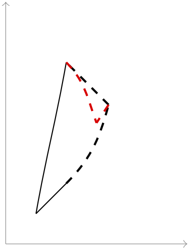
$$R_* = [p]\, \frac{\beta - 1}{2} \begin{pmatrix} \pm 1 \\ -1 \\ 0 \end{pmatrix} + O(|[p]|^3) = O(|[p]|^3).$$

Need for compressions









$$V_t > \max\{V_0, V_\infty\}.$$

- The $\mathcal{R}^P$ (as well as $\mathcal{R}$) decreases whenever a monotone simple wave is split into smaller waves.

Assumption

There is a nonlocal, non-negative Glimm potential $\mathcal{P} = \mathcal{P}(\Gamma^t)$ such that both $\mathcal{P}$ and $\mathcal{V} + \mathcal{P}$ are decreasing,

$$\Delta\mathcal{P} \leq 0 \text{ and } \Delta(\mathcal{V} + \mathcal{P}) \leq 0,$$

across **any** pairwise interaction, and such that $Var(\Gamma^0) + \mathcal{P}(\Gamma^0)$ is bounded.

Theorem

Fix $\epsilon > 0$ and for some $T > 0$, suppose the Assumption holds for $t < T$. Then the scheme can be defined beyond time T; that is, there is some $k \in \mathbb{N}$ such that the k-th interaction time satisfies $t_k > T$.

Proof

- Waves can increase in number when: rarefaction crosses a strong shock, a compressive wave (shock or compression) catches up to a shock, compression collapses.
- Bounded variation implies strong shocks and moderate compressions are finitely many.
- Nonincreasing $\mathcal{P}$ along with some lower bound estimates imply interactions involving strong shocks are finitely many.
- Hence, all eventual interactions are two-in-two-out.
- Leveraging the structure of the p-system, we obtain that the number of interactions are finitely many up to T.

Specific interactions

- Rarefaction crossing a shock:

$$\Delta P \leq -\Delta Var \leq -\Delta\zeta.$$

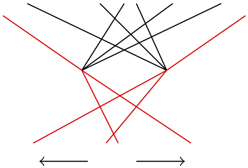
- 0.5ϵ to ϵ can only happen at most $\frac{\mathcal{P}_0}{0.5\epsilon}$ times.
- Compression collapse: We imposed a minimum strength of compression:

$$\Delta P \lesssim -|\zeta_{min}|^3$$

  Hence, finitely many.
- Consequently, all further interactions are two-in-two-out.

Vacuum Wave in Isentropic Gas Dynamics - Lagrangian frame

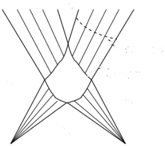
- For $\gamma > 1$, vacuum must be handled in the scheme when working with strong waves, in order to get a large data result.
- Works well in isentropic gas dynamics in Lagrangian frame, since vacuum is stationary in this frame. Looks like an entropy wave in the scheme.
- Along with the states, the width of a vacuum is also tracked.
- Interactions with vacuum



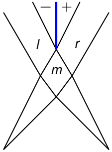
Credits: Alberto Bressan

The idea

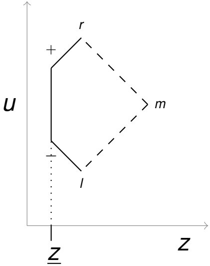
- Idea is to maintain an upper (lower) bound on specific volume (density, pressure) in the scheme, $\underline{p}(\epsilon)$.
- Much more convenient to work in (z, u) with $u \pm z$ are the RIs.
- Working with $\underline{z}$, $\lim_{\epsilon\to 0^+} \underline{z} = 0$.
- Vacuum has $z = \underline{z}$ on both of its sides.
- Strength $\dot{w} = u_+ - u_- =: \Gamma_v$, could be positive (expanding) or negative (contracting)
- A newly formed vacuum starts with zero width and is always expanding.



Young '03

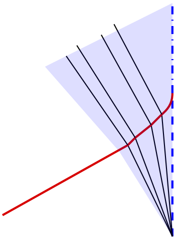


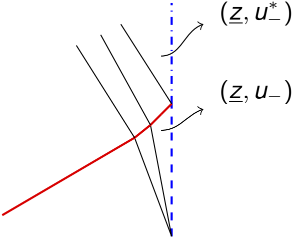
$z_m \geq \underline{z}$,
$z_l + z_r - z_m < \underline{z}$



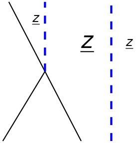
Few interactions

Shocks hitting vacuum.



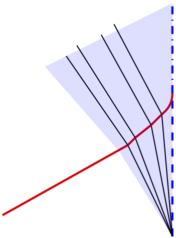


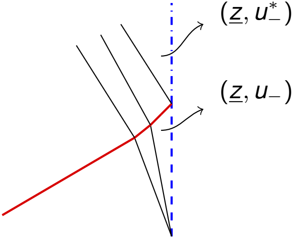
Consecutive vacuums



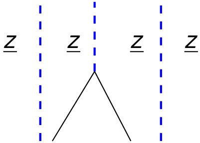
Few interactions

Shocks hitting vacuum.

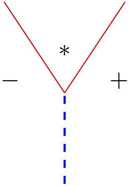


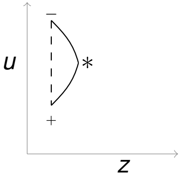


Consecutive vacuums



Vacuum collapse: When $\dot{w} = \Gamma_v < 0$, the width reaches zero in finite time.





Theorem (B-Young '26)

Fix a discretization parameter for the scheme, ϵ. The following holds,

$$z(x,t) \geq \underline{z}(\epsilon),$$

for all (x,t), as long as the scheme continues.