STATEMENT OF RESEARCH

MANAS BHATNAGAR

1. Research Background

My research interests lie in the broader area of Analysis of PDEs (theory and numerics), especially equations that have applications in physics and biology. The questions that intrigue me the most are the structure, properties and visual characteristics of the solutions.

My primary work has been within the field of *Hyperbolic Balance Laws*, the general form of which is,

$$\mathbf{u}_t + \nabla \cdot \mathbf{G}(t, \mathbf{x}, \mathbf{u}) = \mathbf{S}(t, \mathbf{x}, \mathbf{u}).$$

Here, $\mathbf{u} = \mathbf{u}(t, \mathbf{x})$ is the vector of N unknowns and $\mathbf{G}$ is the (smooth) flux matrix. If the source term vector $\mathbf{S}$ is zero, there are no outside forces and $\mathbf{u}$ is conserved, hence, it is then called a *System of Conservation Laws.* The independent variables $t, \mathbf{x} \in \mathbb{R}^m$ are time and space respectively. For physical systems, there is an associated entropy-entropy flux pair. In one-dimensional spatial variable the equations read,

$$\mathbf{u}_t + (\mathbf{F}(\mathbf{u}))_x = \mathbf{S}(t, x, \mathbf{u}). \tag{1.1}$$

The matrix $\mathbf{G}$ is now reduced to a vector $\mathbf{F}$. Initial data $\mathbf{u_0} \in L^\infty$ is given. It called an $N \times N$ *system of hyperbolic balance laws* if for the matrix $D_{\mathbf{u}}\mathbf{F}$, there exist real eigenvalues,

$$\lambda_1(\mathbf{u}) \leq \lambda_2(\mathbf{u}) \leq \ldots \leq \lambda_N(\mathbf{u}), \qquad \forall \mathbf{u}, \tag{1.2}$$

with the corresponding eigenvectors being $\{r_i(\mathbf{u})\}_{i=1}^N$, thereby implying its diagonalizability. A large range of equations of classical physics have this form: fluid and gas dynamics, inviscid combustion equations, thermoelasticity, continuum mechanics, thermomechanics and many more [1, 2, 3]. With certain nuances, for example introducing nonlocal flux and/or source, the equations are even enriched and can model various biological and chemical phenomena, [3, 4], which makes hyperbolic balance laws an interesting and significant field of research.

I am interested in such systems from an analysis point of view. The questions that intrigue me the most are the well-posedness of the equations, that is, existence, uniqueness, and stability of solutions, along with the long-time-behavior and any other special properties.

2. Equations of Gas Dynamics

How gas transforms upon sudden changes in pressure/velocity is of special concern. Propulsion systems (planes, jets, rockets), weather prediction, astrophysics are few of the many fields which heavily rely on the study of gas dynamics.

2.1. **Large amplitude solutions to isentropic gas dynamics.**

As proved in [28], the occurrence of discontinuities in finite time for conservation laws ($N = 2, \mathbf{S} = 0$) is generic. Therefore, the solution leaves the C^1 (continuously differentiable) class of functions even if the initial data is so. Consequently, if we are to look for large-time existence, we need to expand the search

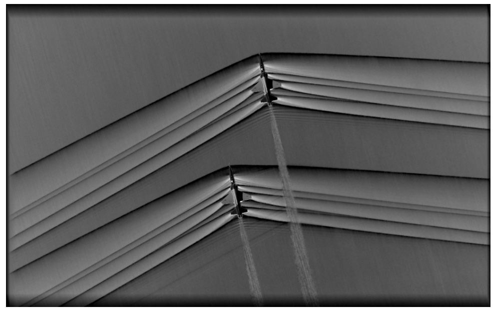

FIGURE 1. Shock waves from planes. Source: NASA

space of solutions. This gives way to weak solutions. A locally bounded measurable function $\mathbf{u}$ is a weak solution to (1.1) on $\mathbb{R} \times [0,T)$ if,

$$\int_0^T \int_{\mathbb{R}} \boldsymbol{\phi}_t \cdot \mathbf{u} + \boldsymbol{\phi}_x \cdot \mathbf{F}(\mathbf{u})\, dxdt + \int_{\mathbb{R}} \boldsymbol{\phi}(x,0) \cdot \mathbf{u}_0(x)\, dx = 0, \tag{2.1}$$

for all test functions $\boldsymbol{\phi}$, with each of its component in $C_c^\infty(\mathbb{R} \times [0,T))$, class of compactly supported and infinitely differentiable functions. One can rightfully expect that the machinery to deal with weak solutions is substantially different from that of classical solutions.

If the inequalities are strict in (1.2), (1.1) is then a strictly hyperbolic system. In [29], Lax studied the case where the fields are either *genuinely nonlinear* ($\forall \mathbf{u}, D\lambda_i \cdot r_i \neq 0$) or *linearly degenerate* ($D\lambda_i \cdot r_i \equiv 0$). Lax proved the existence/uniqueness to a Riemann Problem to (1.1) as long as the two states are sufficiently close. Using these Riemann problems as building blocks, Glimm [30] proved the existence of weak solutions for all time, in the space of functions of Bounded Variation (BV), as long as the initial data is small in the sense of oscillation as well as variation. Glimm constructs a family of approximate solutions (simple functions in between mesh times) with uniformly bounded variation, whose limit (exists using Helly's Selection Theorem) is an entropic weak solution. The method is called the Random Choice Method and it serves the important purpose of keeping discontinuities sharp and waves finite.

There have been several breakthrough contributions, notably the uniqueness, stability of small-amplitude solutions [31, 32, 33, 34, 35, 36], as well as the analysis of more general systems, like the ones that have fields that are *not genuinely nonlinear*, [37, 38, 39] and nonstrictly hyperbolic systems, [40].

Recently, Glimm's Random Choice Method of constructing such approximate solutions has phased out to a more popular method: The *Front Tracking (FT) Method*, [41, 42, 43, 44, 45], wherein each discontinuity is tracked until two of them meet, resulting in a Riemann Problem. The gain is that the scheme carries more information and therefore, is conducive to stability analysis but with a downside that one has to control the number of waves (avoid infinitely many waves in finite time).

Most of the substantial progress that has been made for (1.1) ($\mathbf{S} = 0, m = 1$ and $N \geq 2$) requires the assumption of small amplitude. This assumption is very restrictive since weak shocks are seldom seen in nature. The magnitude of shock waves in Figure 1 is substantially huge compared to the ones for which the existence, uniqueness theory exists in the current literature. Although L^∞ solutions to 2×2 systems obtained through compensated compactness methods are known, [46, 32], they exhibit little knowledge about the structure of the solutions. The existence of general BV solutions with large amplitudes to the system as basic as the p-system (isentropic gas dynamics),

$$\begin{aligned} \tau_t - v_x &= 0, \\ v_t + p(\tau)_x &= 0. \end{aligned} \tag{2.2}$$

is not known. Here, the pressure p is a function of the specific volume τ, which follows the relation $p(\tau) = \tau^{-\gamma}$ in case of gamma-law. v is the velocity and $\gamma \geq 1$ is intrinsic to the gas, which is (loosely) a measure of the stiffness. The physical significance of this system renders the existence/uniqueness question quite important.

One of our (with Robin Young) current projects is to analyze solutions of (2.2) with strong shocks, going up to the order as in Figure 1. The equations as above are in the material (Lagrangian) coordinates. Existence of large amplitude solutions to the isothermal ($\gamma = 1$) case is known, thanks to Nishida, [47]. In this scenario, the wave curves are translation invariant and there is no vacuum, which makes the analysis easier compared to the $\gamma > 1$ case. In [48], Nishida and Smoller obtained global existence for (2.2) assuming $\gamma - 1$ is sufficiently small. This smallness condition has been made more precise very recently in [49]. Temple class systems [50] and systems satisfying Bakhvalov conditions [51] are other special examples for which general BV solutions are known.

To obtain large amplitude solutions for (2.2), we figured that one must need more accurate representation of waves in the FT method. Therefore, we constructed *The Modified Front Tracking (mFT) Scheme* [16]. We developed the novel idea of *composite waves*. Moreover, we use *compression curves* as well to solve Riemann Problems, which we augment as the 'Extended Riemann Problem' (xRP).

To construct approximate solutions, we follow the philosophy of keeping states exact and not using any nonphysical waves. As a result, we introduce the concept of composite waves (needed only for general systems for $N > 2$ and not in (2.2)) to avoid infinitely many waves in finite time. The preprint version has been made available online [16]. In our scheme, we introduce consistent procedures to split rarefactions that become too wide or too strong and procedures for simple waves with nonzero widths to interact and generate waves with widths. Moreover, we prove that the existence of BV weak solutions for (1.1) with (1.2) solely boils down to obtaining uniform BV estimates for approximate solutions through our scheme. To do this, we work with weak* solutions introduced in [52], wherein the authors also proved the equivalence of weak* and weak solutions. We use Banach-Alaoglu to prove convergence instead of Helly's Selection Theorem, since the FT solutions we construct lie in a dual of a space $(L^\infty([0,T];\mathcal{M}_{loc}) = L^1([0,T];C)^*)$. This has an ostensible advantage because the latter cannot be applied in multi-D whereas the former has no such limitations.

Theorem 2.1 ([16],Convergence of Scheme for general systems)**.** *Under some mild assumptions, the Modified FT algorithm can be continued up to all times with output,*

$$U^\epsilon \in W^{1,\infty}([0,T];\mathcal{M}_{loc}), \qquad U^\epsilon(\cdot,t) \in BV_{loc}.$$

Fix a time $T > 0$. Suppose for all ϵ sufficiently small, $Var(U^\epsilon(\cdot,t)) \leq M$ for $t \in [0,T]$, where M is a constant. Then U^ϵ converges (in weak) to U, an entropic weak solution.*

We are currently working on a Neural Network implementation of the scheme and hence, have not open-sourced the code yet. An example of a simulation for a 2×2 system without the NN component is in Figure 2.

3. Weakly Hyperbolic System of Balance Laws

If in (1.2), $\lambda_i(\mathbf{u}) = \lambda(\mathbf{u})$ for all i's (all eignevalues of $D\mathbf{F}$ coalesce), it is a weakly hyperbolic system. When charge flows in a semiconductor or a dying star collapses, the pressure forces

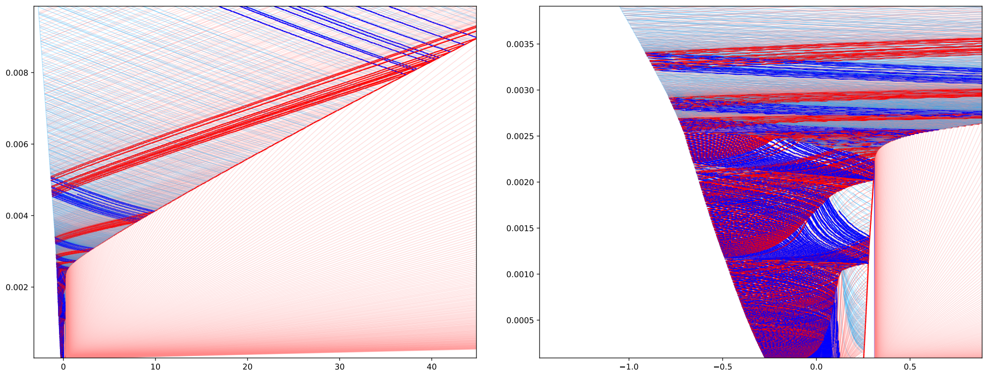

FIGURE 2. FT simulation on the left, with a strip near starting time zoomed in on the right.

are negligible and the repulsive (force between electrons) or attractive (self-gravitation in dying stars) forces dominate. The absence of pressure in such scenarios leads to information propagation only along a single characteristic field. An example of such a system is the pressureless Euler-Poisson-alignment (EPA) system,

$$\begin{aligned} &\rho_t + \nabla\cdot(\rho\mathbf{v}) = 0,\\ &\mathbf{v}_t + (\mathbf{v}\cdot\nabla)\mathbf{v} + \nu\mathbf{v} = -k\nabla\phi + \psi*(\rho\mathbf{v}) - \mathbf{v}\psi*\rho,\\ &-\Delta\phi = \rho - c. \end{aligned} \tag{3.1}$$

with $\rho, \mathbf{v}$ being density and velocity, the spatial variable $\mathbf{x}\in\mathbb{R}^m$ (or $\mathbb{T}^m$, m-dimensional torus), ν is the constant damping, ψ is the influence function, c is the background and the sign of k determines the type of force between the particles (attractive or repulsive). For various class of influence functions and parameter values, these equations model charge flow in semiconductors, self-gravitation in stars and nebulae, flocking of birds and other animals, [5, 4, 6, 7].

I am interested in these equations from the *Critical Threshold (CT) Phenomena* perspective. To motivate the CT Phenomena, consider, for example, the 1D inviscid Burgers' equation $u_t + uu_x = 0$. Straightforward calculations show that an initially smooth solution develops singularity in finite time if the gradient of initial data is negative somewhere. Therefore, the set of initial data, for which the solution is smooth for all time, is trivial. This points to the fact that the nonlinearity (or the convection term in this case) leads to discontinuities, as opposed to the solutions of the transport equation (constant speed) wherein smoothness persists for all time. However, this bad term can be offset with the addition of good source term(s). The dichotomy between the good and bad terms is a topic of interest. In stark contrast to the inviscid Burgers', it might be that the good term annihilates the bad term and the solution remains smooth for all time for any initial data. This is the case for the viscous Burgers' as well as fractional Burgers' ($u_t + uu_x = -(-\Delta)^\alpha u$, $\alpha\geq 1/2$ and $\alpha = 1$ is the viscous case), [8]. Similar is the case for (3.1) with strongly singular influence function, [9, 10]. The (fractional) dissipative term dominates the convection term leading to the persistence of smoothness.

CT Phenomena occurs when there is a balance between the good and bad terms. So, the space of initial data is divided into exactly two regions, subcritical and supercritical. If the initial data lies completely in the subcritical region, then the smoothness persists for all time. On the other hand, if the initial data has a nonempty intersection with the supercritical region, then discontinuities develop in finite time. It was first introduced in [15] in the context of Euler-Poisson (EP) systems.

In the presence of such dichotomy, the classical theory for the existence of smooth solutions is limited, in the sense that not only is it applicable for short times, [11, 12], but also the class of initial data for which global-in-time solution can be shown is 'small', see [13, 14]. In classical theory, we need an assumption of 'smallness' in some sense on the set of initial data that can be considered. Therefore, such analysis does not give any information about a 'large' class of initial data. This is where CT Analysis kicks in as it goes beyond the techniques of classical methods that only show existence of global-in-time solutions for 'small' class of initial data, and short times.

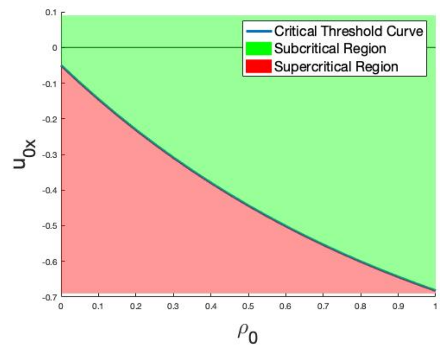

FIGURE 3. Example of subcritical, supercritical regions with the critical threshold curve for $m = 1, \psi \equiv 0, c > 0$ in (3.1). The graph is in the phase space of initial data.

We (along with Hailiang Liu) analyzed various forms of (3.1) in one dimensional spatial variable. In [19], we introduced a new technique to analyze CTs by introducing a singular transformation to obtain a linear ODE system, and then perform a phase plane analysis. Our developed methodology found a special mention in the recent work [25]. We use this method in our subsequent work [20], to obtain bounds on subcritical/supercritical regions for EP systems with background that varies spatially. To my knowledge, this is the first such result for EP systems with variable background. In our next series of works [21, 22], we introduce a new system that serves as a physical extension to the one studied by Liu, Tadmor in [26]. We study the local as well as nonlocal version and prove the existence/uniqueness of solutions along with performing a detailed CT analysis. We then discovered a new method to derive precise thresholds and strengthen density bounds in the EPA system with weakly singular influence function [23] that had been studied by Tan in [27]. In our (with Liu and Changhui Tan) more recent work, we construct subcritical/supercritical by piecing together solution of ODEs in the phase space. Some of our important results are included in Section 3.1 and 4. The complete list of publications/preprints is [17, 18, 19, 20, 21, 22, 23, 24].

3.1. **Spherically symmetric Euler-Poisson systems with nonzero background and repulsive forcing.** This work is in collaboration with Dr Hailiang Liu, [17].

In [53], Tan obtains critical thresholds for the Euler-Poisson system ($\psi \equiv 0$ in (3.1)) under the radial symmetry assumption, $\rho = \rho(r,t), \mathbf{v} = v(r,t)\frac{\mathbf{x}}{r}$, and with $k < 0$. A contemporary advancement for this system is by Carrillo and Shu, [54], where the authors show that for $m \neq 4$, the subcritical region is trivial or zero measure in the phase space. The authors show that if initial data is nonconstant on an interval, then the trajectories must run into each other. Their method and analysis only applies to the case when $c > 0$ and does not give any idea for $m = 4$. Rozanova also showed similar results, [55].

We derive the subcritical/supercritical regions for $(c \geq 0, k > 0)$ in all dimensions by building upon the ideas in [53]. We find a new quantity that follows much simpler ODE dynamics (along characteristics). A thorough analysis of this quantity gives the regions. Even though the limit $c \to 0^+$ is singular, our analysis is general and applies to both, which, to my knowledge, is quite novel as all other existing analysis in the literature are specific to a single case. In particular, for $c = 0$, we derive the precise CT curve, without any assumption on the sign of velocity. The best known results for the $c = 0$ case is [56], wherein velocity is assumed to be strictly positive and [53], where the CT curve is not precise. Following is our result for $m \geq 3$.

Theorem 3.1 ([17], CT in radially symmetric EP system). *Suppose $c = 0, \nu = 0, m \geq 3, \psi \equiv 0$ in (3.1) subject to C^1, radially symmetric initial data. If for all $r > 0$, the set of points,*

$$(r, u_0(r), \phi_{0r}(r), u_{0r}(r), \rho_0(r)) \in \Sigma_m \cup \left\{(\alpha, x, y, z, 0) : x > 0, z \geq -\frac{ky}{x}\right\},$$

where $\Sigma_m \subseteq \left\{(\alpha, x, y, z, \omega) : y < 0, \omega > 0,\ \frac{xz+ky}{\alpha\omega} > -\frac{k}{m-2}\right\}$ (Σ_m defined via evaluation of solution of a differential equation with initial data as parameters), then there is global solution.

Moreover, if there exists an $r_c > 0$ such that

$$(r_c, u_0(r_c), \phi_{0r}(r_c), u_{0r}(r_c), \rho_0(r_c)) \notin \Sigma_m \cup \left\{(\alpha, x, y, z, 0) : x > 0, z \geq -\frac{ky}{x}\right\},$$

then there is finite-time-breakdown.

4. Other Important Contributions

By introducing a new methodology, we achieved the following. Only the more interesting result for the *weak damping* case is stated.

Theorem 4.1 ([19]). *Consider (3.1) with $m = 1, \psi \equiv 0, k > 0$ and a constant background c subject to C^1 initial data. Assume the weak damping ($\nu < 2\sqrt{kc}$) regime. Then there exists a unique solution $\rho, u \in C^1(\mathbb{R} \times (0, \infty))$ iff $\forall x \in \mathbb{R}$,*

$$(\rho_0(x), u_{0x}(x)) \in \{(\rho, d) : -\rho Q_1(1/\rho) < d < -\rho Q_2(s^* - 1/\rho),\ \rho \in (1/s^*, \infty)\},$$

where $s^* > 0$ is uniquely determined, and $Q_1 : [0, s^*] \longrightarrow \mathbb{R}^+ \cup \{0\}$ is a continuous function satisfying

$$\frac{dQ_1}{ds} = \nu + \frac{k}{Q_1}(1 - cs),\ Q_1(0) = 0,$$

and $Q_2 : [0, s^*] \longrightarrow \mathbb{R}^- \cup \{0\}$ is another continuous function satisfying

$$\frac{dQ_2}{ds} = -\nu + \frac{k}{Q_2}(c(s^* - s) - 1), \quad Q_2(0) = 0.$$

Following is our result for the variable background case, which is the first result with this consideration. The bounds on subcritical region are stated.

Theorem 4.2 ([20]). *Consider (3.1) with $m = 1, \psi \in L^\infty, k < 0$ and a smooth background $c = c(x) \in L^\infty$ subject to C^1 initial data on a torus $\mathbb{T}$. Let $0 < c_1 = \min_{x \in \mathbb{T}} c(x)$, $c_2 = \max_{x \in \mathbb{T}} c(x)$, $\psi_M = \max_{x \in \mathbb{R}} \psi(x)$, $\psi_m = \min_{x \in \mathbb{R}} \psi(x)$. If*

$$u_{0x}(x) > \frac{\lambda_M \rho_0(x)}{c_1} - \Theta_M - \nu - \psi * \rho_0(x) \quad \forall x \in \mathbb{T},$$

then there exists a unique global solution $\rho, u \in C^1((0, \infty) \times \mathbb{T})$. Here, λ_M, Θ_M are both explicit functions of c_1, c_2, ψ_M, ψ_m.

The above result is with $k<0$ (attractive forcing), which is relatively easier. In our more recent work, we obtain results for $k>0$ in EPA. The bounds on subcritical region for the *weak alignment* regime are stated in the following result.

Theorem 4.3 ([18]). *Consider* (3.1) *with* $m=1, k>0, \nu=0$, *constant background* c *and* $\psi \in L^\infty$ *with* $\psi_m \le \psi \le \psi_M$, *subject to* C^1 *initial data on a torus* $\mathbb{T}$. *Also assume the weak alignment* ($\psi_M < 2\sqrt{k/c}$) *regime. Then under the admissible condition*

$$\psi_M - \psi_m < \frac{e^{\frac{\tan^{-1}\hat{z}}{\hat{z}}}\left(1-e^{-\frac{\pi}{\hat{z}}-\frac{\pi}{\tilde{z}}}\right)}{2\left(1+e^{-\frac{\pi}{\tilde{z}}}\right)} 2\sqrt{k/c}, \tag{4.1}$$

if the initial data lie in the region Σ, *namely*

$$\big(u_{0x} + \psi * \rho_0(x), \rho_0(x)\big) \in \Sigma, \quad \forall\, x \in \mathbb{T},$$

there is a global (in time) smooth solution. Here, the parameters $\hat{z}$ *and* $\tilde{z}$ *are defined as*

$$\hat{z} := \sqrt{\left(\frac{2\sqrt{k/c}}{\psi_M}\right)^2 - 1} \quad \text{and} \quad \tilde{z} := \sqrt{\left(\frac{2\sqrt{k/c}}{\psi_m}\right)^2 - 1}. \tag{4.2}$$

The region Σ *is a subset of* $\mathbb{R}\times\mathbb{R}_+$, *defined explicitly by piecing together three trajectories of a linear system of* 2×2 *ODEs, for different parameters.*

5. Ongoing and Future Projects

5.1. **Implementation of the scheme developed.** (With Zachary Williams and Young) As a part of an undergraduate thesis project (for Zachary), we are in the process of refining the code that would result in the implementation of our scheme. Hence, we would be solving Extended Riemann Problems instead of the usual Riemann Problems at each interactions, along with keeping track widths of rarefactions and compressions.

In our scheme, the wavespeeds are obtained through least squares. This choice is specifically to minimize the residual on a single wave (or discontinuity). However, if one considers the complete scheme into account, then it is a high-dimensional minimization problem, where the dimension is equal to the number of waves in the scheme. The residual would be a composition of numerous (activation) functions. We would like to leverage this aspect and incorporate machine learning techniques to find the optimum wavespeeds for each wave, to hope for faster convergence.

Not only this, there are free parameters within the initialization of the scheme as well that have to be chosen. Hence, the scheme offers a lot of freedom and we would like to take advantage of that.

This project has a lot of scope for students, especially those having a knack for coding. We only have a running code for 2×2 system. We plan to extend it to systems such as equations of Magnetohydrodynamics and String. That requires a substantial modification of the code and rewriting the solvers, which would be a great project to deeply understand Conservation Laws as well as working with and handling complex code repositories.

5.2. **CT analysis in EP equations with pressure.** Moving forward from (3.1), it is only natural to ask the behavior of the equations with regard to CT if there is a pressure term, making the system strictly hyperbolic. In [57], Tadmor and Wei obtained bounds on subcritical/supercritical regions with adiabatic pressure and zero background. They use an approach similar to Lax [28] to reduce the system to two ODEs along the two characteristics and then use an invariant region argument. However, once a nonzero background is included, things become quite complicated. There is a global existence result by Guo, Han, Zhang [13], with initial data in a small neighborhood around the equilibrium solution. However, a general CT result is still absent in literature. I have multiple ideas to tackle this problem and would be working on this in the near future. This project would be highly suitable for a student interested in working with PDEs/ODEs.

5.3. **CT analysis for** (3.1) **with** $m \geq 2$**.** In higher spatial dimensions, we need to control the gradient matrix $\nabla \mathbf{u}$ in L^∞ as opposed to u_x in one dimension. In [58], He and Tadmor analyzed (3.1) with $m = 2, k = 0$, that is, the EA system. To my knowledge, this is the most general result for CTs with $m = 2$ in (3.1). Hence, several aspects still remain unaddressed, the first step being the pressureless EP/EPA system in 2-D. This is one of my important future projects, which has a substantial component with prospective students.

5.4. **Stability of large-amplitude-solutions.** (With Dr Geng Chen and Young) Recently, Chen, Krupa, Vasseur [35] proved the L^2 stability for the isentropic gas dynamics for weak waves, consequently getting rid of past assumptions required for uniqueness, [45]. They used the relative entropy method to derive comparisons with smoother solutions.

Once the existence of large amplitude solutions to (2.2) is known, it is natural that the same questions arise. With Chen, we intend to work on analogous results in the case when strong shocks are present.

5.5. (2.2) **with vacuum.** (With Young) We are working towards extending our Modified Front Tracking Scheme to include a fourth type of wave, namely vacuum, and also obtain uniform (in some sense) bounds on variation.

In [59], Miroshnikov and Young analyzed the 'Extended Riemann Problem' involving vacuum. In a future work, we plan to consistently apply this to our FT scheme. To remove the primary roadblock of vacuum formation in the scheme, we plan to include vacuum waves into the scheme itself, which obviously comes at a cost. To obtain a functional that has nonincreasing variation at each interaction becomes extremely complicated. In particular, there are certain special interactions that need to be addressed. Obtaining bounds here would be the main (and difficult) step as the equations have parabolic degeneracy at vacuum.

5.6. **Smooth periodic solutions to the relativistic Euler.** (With Christopher Alexander, Geng Chen, Yannan Shen, Blake Temple and Young, as a part of the SQuaREs program) Very recently, Temple and Young proved the existence of smooth, space and time periodic, oscillatory solutions of the 3×3 compressible Euler equations in one space dimension, [60]. We did some preliminary analysis on the relativistic Euler equations in flat space. It seems that these equations also exhibit such solutions. Using the techniques in [60], we plan to prove smooth, periodic solutions to these equations.

References

[1] D. Serre. Systems of Conservation Laws 1: Hyperbolicity, Entropies, Shock Waves. *Cambridge University Press*, 1999.

[2] C.M. Dafermos. Hyperbolic Conservation Laws in Continuum Physics. *Springer-Verlag Berlin Heidelberg*, Vol. 325, 2010 (3rd Ed.).

[3] G. -B. Whitham. Linear and Nonlinear Waves. *Pure and Applied Mathematics, Wiley-Interscience*, 1999.

[4] P.A. Markowich, C. Ringhofer and C. Schmeiser. Semiconductor Equations. *Springer-Verlag*, Berlin, Heidelberg, New York, 1990.

[5] T. Makino. On a Local Existence Theorem for the Evolution Equation of Gaseous Stars. *Studies in Mathematics and its applications*, 18:459–479, 1986.

[6] F. Cucker and S. Smale. Emergent Behavior in flocks. *IEEE Transactions on Automatic Control*, 52(5): 852–862, 2007.

[7] F. Cucker and S. Smale. On the mathematics of emergence. *Japanese Journal of Mathematics*, 2(1): 197–227, 2007.

[8] A. Kiselev, F. Nazarov, R. Shterenberg. Blow Up and Regularity for Fractal Burgers Equation. *Dynamics of PDE*, 5(3):211–240, 2008.

[9] T. Do, A. Kiselev, L. Ryzhik, C. Tan. Global regularity for the fractional Euler alignment system. *Arch. Rat. Mech. Anal.*, 228(1): 1–37, 2018.

[10] A. Kiselev, C. Tan. Global regularity for 1D Eulerian dynamics with singular interaction forces. *SIAM J. Math. Anal.*, 50(6):6208–6229, 2018.

[11] T. Kato. The Cauchy problem for quasi-linear symmetric hyperbolic systems. *Arch. Rational Mech. Anal.*, 58: 181–205, 1975.

[12] A. Majda. Compressible Fluid Flow and Systems of Conservation Laws in Several Space Variables. *Springer Science+Business Media*, Vol. 53, 1984.

[13] Y. Guo, L. Han and J. Zhang. Absence of shocks for one dimensional Euler–Poisson system. *Arch. Rational Mech. Anal.*, 223: 1057–1121, 2017.

[14] S. Kawashima, W.-A. Yong. Decay estimates for hyperbolic balance laws. *Journal for Analysis and its Applications*, 28:1–33, 2009.

[15] S. Engelberg, H. Liu, E. Tadmor. Critical thresholds in Euler-Poisson equations. *Indiana University Math. Journal*, 50:109–157, 2001.

[16] M. Bhatnagar, R. Young. Weak* Solutions III: A Convergent Front Tracking Scheme. *ArXiv preprint*, `https://arxiv.org/abs/2411.09086`

[17] M. Bhatnagar, H. Liu. A complete characterization of sharp thresholds to spherically symmetric multidimensional pressureless Euler-Poisson systems. *ArXiv preprint*, `https://arxiv.org/abs/2302.04428`

[18] M. Bhatnagar, H. Liu and C. Tan. Critical Thresholds in the Euler-Poisson-alignment system. *Journal of Differential Equations*, 375: 82–119, 2023.

[19] M. Bhatnagar, H. Liu. Critical thresholds in one-dimensional damped Euler-Poisson systems. *Math. Mod. Meth. Appl. Sci.*, 30(5):891–916, 2020.

[20] M. Bhatnagar, H. Liu. Critical threshold phenomena in 1D pressureless Euler-Poisson systems with variable background. *Physica D: Nonlinear Phenomena*, 414, 2020.

[21] M. Bhatnagar, H. Liu. Well-posedness and critical thresholds in nonlocal Euler system with relaxation. *Disc. Cont. Dyn. Sys. A*, 41(11): 5271–5289, 2021.

[22] M. Bhatnagar, H. Liu. Sharp critical thresholds in a hyperbolic system with relaxation. *Disc. Cont. Dyn. Sys. A*, 41(12): 5851–5869, 2021.

[23] M. Bhatnagar, H. Liu. Global dynamics of the Euler-alignment system with weakly singular kernel. arXiv: 2110.10314, 2021.

[24] M. Bhatnagar, H. Liu. On Critical Thresholds for Hyperbolic Balance Law Systems. *Hyperbolic Problems: Theory, Numerics, Applications (Conference)*, 1: 145–155, 2024.

[25] Y.-P. Choi, D.-H. Kim, D. Koo, E. Tadmor. Critical thresholds in pressureless Euler–Poisson equations with background states. *ArXiv: 2402.12839*, `https://arxiv.org/abs/2402.12839`, 2024.

[26] H. Liu, E. Tadmor. Critical Thresholds in a Convolution Model for Nonlinear Conservation Laws. *SIAM J. Math. Anal.*, 33(4): 930–945, 2001.

[27] C. Tan. On the Euler-alignment system with weakly singular communication weights. *Nonlinearity*, 33: 1907–1924, 2020.

[28] P. Lax. Development of singularities of solutions of nonlinear hyperbolic partial differential equations. *Journal of Math. Phys.*, 5, 611–613, 1964.

[29] P. Lax. Hyperbolic Systems of Conservation Laws II. *Comm. on Pure and Applied Math.*, 10, 537–566, 1957.

[30] J. Glimm. Solutions in the Large for Nonlinear Hyperbolic Systems of Equations. *Comm. on Pure and Applied Math.*, 18, 697–715, 1965.

[31] J. Glimm, P. Lax. Decay of solutins of systems of nonlinear hyperbolic conservation laws. *Memoirs of the AMS*, 101, 1970.

[32] R. -J. DiPerna . Convergence of approximate solutions to conservation laws. *Arch. for Rational Mech. and Anal.*, 82, 27–70, 1983.

[33] R. Young. Sup-norm stability for Glimm's scheme. *Comm. Pure Appl. Math.*, 46(6), 903–948, 1993.

[34] S. Bianchini, A. Bressan. Vanishing viscosity solutions of nonlinear hyperbolic systems. *Ann. of Math.*, 161, 223–342, 2005.

[35] G. Chen, S. Krupa and A. Vasseur. Uniqueness and Weak-BV Stability for 2×2 Conservation Laws. *Arch. for Rational Mech. and Anal.*, 246, 299–332, 2022.

[36] A. Bressan, G. Guerra. Unique solutions to hyperbolic conservation laws with a strictly convex entropy. *J. of Differential Equations*, 387, 432–447, 2024.

[37] T.-P. Liu. The Riemann Problem for General Systems of Conservation Laws. *J. of Differential Equations*, 18, 218–234, 1975.
[38] T.-P. Liu. Admissible solutions of hyperbolic conservation laws. *Memoirs of the AMS*, 30(240), 1975.
[39] T. -P. Liu, T. Ruggeri. Entropy Production and Admissibility of Shocks. *Acta Mathematicae Applicatae Sinica*, 19(1), 1–12, 2003.
[40] B. Keyfitz, H. -C. Kranzer. Entropy Production and Admissibility of Shocks. *Arch. for Rational Mech. and Anal.*, 72, 219–241, 1980.
[41] R. -J. DiPerna. Global existence of solutions to nonlinear hyperbolic systems of conservation laws. *J. of Differential Equations*, 20, 187–212, 1976.
[42] N. -H. Risebro. A front-tracking alternative to the random choice method. *Proc. Amer. Math. Soc.*, 117, 1125–1139, 1993.
[43] A. Bressan. Global solutions to systems of conservation laws by wavefront tracking. *J. Math. Anal. App.*, 170, 413–432, 1992.
[44] P Baiti, H. -K. Jenssen. On the Front-Tracking Algorithm. *J. Math. Anal. App.*, 217, 395–404, 1998.
[45] A. Bressan. Hyperbolic Systems of Conservation Laws: the one dimensional Cauchy problem. *Oxford Lecture Series in Math. and its Applications*, 20, 2000.
[46] P. -L. Lions, B. Perthame and E. Tadmor. Kinetic Formulation of the Isentropic Gas Dynamics and p-Systems. *Comm. Math. Phys.*, 163, 415–431, 1994.
[47] T. Nishida. Global Solution for an Initial Boundary Value Problem of a Quasilinear Hyperbolic System. *Proc. Japan Acad.*, 44, 642–646, 1968.
[48] T. Nishida, J. -A. Smoller. Solutions in the Large for Some Nonlinear Hyperbolic Conservation Laws. *Comm. Pure App. Math.*, 26, 183–200, 1973.
[49] F. Wu, Z. Wang and F. Chen The Global Existence of BV Solutions of the Isentropic p-System with Large Initial Data. *Acta Mathematica Scientia*, 43, 1668–1674, 2023.
[50] B. Temple. Systems of conservation laws with invariant submanifolds. *Trans. Amer. Math. Soc.*, 280, 781–795, 1983.
[51] N. -S. Bakhvalov. The existence in the large of a regular solution of a quasilinear hyperbolic system. *Z. Vycisl. Mat. i Mat. Fiz.*, 10, 969–980, 1970 (in Russian).
[52] A. Miroshnikov, R. Young. Weak* solutions: A new perspective on solutions to systems of conservation laws. *Methods and Applications of Analysis*, 24(3), 351–382, 2017.
[53] C. Tan. Eulerian dynamics in multi-dimensions with radial symmetry. *SIAM J. Math. Anal.*, 53(3): 3040–3071, 2021.
[54] J. -A. Carrillo, R. Shu. Existence of radial global smooth solutions to the pressureless Euler-Poisson equations with quadratic confinement. *Arch. for Rational Mech. and Anal.*, 247(73), 2023.
[55] O. Rozanova. On the behavior of multidimensional radially symmetric solutions of the repulsive Euler–Poisson equations. *Physica D: Nonlinear Phenomena*, 443: 133578, 2023.
[56] D. Wei, E. Tadmor, H. Bae. Critical Thresholds in multi-dimensional Euler-Poisson equations with radial symmetry. *Commun. Math. Sci.*, 10(1):75–86, 2012.
[57] E. Tadmor, D. Wei. On the global regularity of subcritical Euler-Poisson equations with pressure. *J. Eur. Math. Soc.*, 10:757–769, 2008.
[58] S. He, E. Tadmor Global regularity of two-dimensional flocking hydrodynamics. *C. R. Math.*, 355(7): 795–805, 2017.
[59] A. Miroshnikov, R. Young. Weak* Solutions II: The Vacuum in Lagrangian Gas Dynamics. *SIAM J. Math. Analysis*, 39(3), 1810–1843, 2017.
[60] B. Temple, R. Young. The Nonlinear Theory of Sound. *ArXiv preprint*, `https://arxiv.org/abs/2305.15623`, 2023.

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